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MAHLER MEASURE EVALUATIONS IN TERMS OF POLYLOGARITHMS

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MAHLER MEASURE EVALUATIONS IN TERMS OF POLYLOGARITHMS

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To Mom, Dad, Sister and Grandma.

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We prove a conjecture of Boyd by showing that the logarithmic Mahler measure of a certain integer polynomial in three variables is equal to $\frac{28}{5\pi^2}\zeta(3)$. The proof proceeds by expressing the Mahler measure as a combination of integrals of one-variable logarithmic forms, evaluating these in terms of polylogarithm functions at algebraic arguments, and using identities to simplify the expression.

Next, we indicate how the techniques used in the previous example can be applied to give Mahler measure evaluations in terms of polylogarithms for two families of three variable polynomials. As an example, we work out the details for a four-parameter subfamily.

Finally, we discuss an alternative, more algebraic approach to this sort of calculation. This method, developed by Rodríguez Villegas, Boyd, Maillot and others, relies on showing that certain elements of algebraic K-groups are

equal to zero. We reinterpret our original problem in this context and consider attempts at its resolution.

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Chapter 1

Introduction

For a nonzero Laurent polynomial $P(x_1, \ldots, x_n) \in \mathbb{C}[x_1{}^{\pm 1}, \ldots, x_n{}^{\pm 1}]$, the *logarithmic Mahler measure*[1] of P is defined as

$$m(P) = \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log\bigl|P(x_1, \ldots, x_n)\bigr| \, \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}, \tag{1.1}$$

where $\mathbb{T}^n$ is the n-torus $\bigl\{(x_1, \ldots, x_n) \in \mathbb{C}^n \,:\, |x_1| = \cdots = |x_n| = 1\bigr\}$. When P is a polynomial in one variable, Jensen's formula allows us to rephrase this in terms of the roots of $P(x)$ outside the unit circle:

$$m(P) \;=\; \log|a| + \sum_{j=1}^{d} \log^+|\alpha| \tag{1.2}$$

where $P(x) = a \prod_{j=1}^{d} (x - \alpha_j)$ and $\log^+(x) = \log\bigl(\max\{x, 1\}\bigr)$.

The motivations for studying Mahler measure have evolved with time. Chronologically, (1.2) came first; D. H. Lehmer [14] examined this quantity in the 30s in an effort to optimize a technique due to Pierce for finding large primes. Mahler [16] introduced (1.1) in the 60s as a generalization of (1.2)

[1] We will refer to $m(P)$ simply as the Mahler measure of P, although this name usually refers to the quantity $M(P) = \exp\bigl(m(P)\bigr)$, the geometric mean of $|P|$ on $\mathbb{T}^n$.

to polynomials of several variables; he used it as a tool for giving a simplified proof of the Gelfond-Mahler inequality.

Over the years, interest grew in a number of questions more intrinsic to Mahler measure. Perhaps the most famous of these is Lehmer's problem, which concerns lower bounds on $m(P)$ for $P \in \mathbb{Z}[x]$. But in the 80s, Smyth [21] discovered the following identities:

$$m(1+x+y) \;=\; L'(\chi,-1),$$

where χ is the Dirichlet character of conductor 3, and

$$m(1+x+y+z) \;=\; \frac{7}{2\pi^2}\zeta(3).$$

These amazing relations and others like them raised interest not just in inequalities satisfied by $m(P)$, but in equalities—in the particular values obtained by $m(P)$, particularly for P with integer coefficients. The appearance of zeta and L-functions linked Mahler measure to some of the more important and mysterious objects in modern number theory.

Since then, much has been proven and conjectured about special values of $m(P)$. In particular, one of our main results is the proof of the following conjecture of Boyd:

$$m\big(1+x+(1-x)(y+z)\big) = \frac{28}{5\pi^2}\,\zeta(3). \tag{1.3}$$

In chapter 2, we supply some background, giving the needed facts about Mahler measure and about polylogarithms, the central tool in all of our calculations.

In chapter 3, the proof of (1.3) is given. A theorem of Lalín is used to change the original (three-variable) integral into a one-variable integral. This is then decomposed as a combination of integrals of certain logarithmic forms, which may be evaluated using polylogarithms. A number of identities are then used to simplify the resulting expression down to a single term.

Chapter 4 explores generalizations of the techniques used to prove (1.3). The logarithmic forms used in chapter 3 are examined more carefully. Descriptions are then given of how to evaluate the Mahler measure for two families of rational functions, and it is shown that these evaluations are always "homogeneous" linear combinations of polylogarithm values with algebraic arguments. Details are provided for one subfamily.

Chapter 5 examines an algebraic approach to these Mahler measure calculations. It is shown how conditions for success with this method may be interpreted in terms of algebraic K-theory. We discuss the prospects for proving (1.3) using these techniques.

Chapter 2

Preliminaries

2.1 Mahler measure

The first natural question to ask about Mahler measure is if it is even finite. The definition given in (1.1) may be rephrased as follows:

$$m(P) = \int_0^1 \cdots \int_0^1 \log\bigl|P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n})\bigr| \, d\theta_1 \cdots d\theta_n \tag{2.1}$$

The integrand tends to $-\infty$ as $(\theta_1, \ldots, \theta_n)$ approaches points belonging to $\{P(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}) = 0\}$. But the integral does indeed converge to a finite real number, regardless of whether P vanishes on $\mathbb{T}^n$ or not; Mahler actually provided lower bounds for $m(P)$ in [16]).

An obvious but important observation is that:

$$m(PQ) = m(P) + m(Q). \tag{2.2}$$

We may in fact also define Mahler measure for rational functions—either by using $m(P/Q) = m(P) - m(Q)$ or simply by allowing rational functions in (1.1)—and this too will be well-defined and finite. Henceforth, P will be allowed to be a rational function in $\mathbb{C}(x_1, \ldots, x_n)$ unless otherwise specified.

The following propositions describe some of the modifications that may be made to P that leave its Mahler measure unchanged.

Proposition 2.1. *For any* $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ *with* $|\lambda_1| = \cdots = |\lambda_n| = 1$,

$$m\big(P(\lambda_1 x_1, \ldots, \lambda_n x_n)\big) = m\big(P(x_1, \ldots, x_n)\big)$$

This holds because the change of variables $x_j \to \lambda_j x_j$ simply rotates the circle $|x_j| = 1$.

To interpret the next proposition, we think of the monomial $x_1^{m_1} \cdots x_n^{m_n}$ as corresponding to the vector $(m_1, \ldots, m_n)^T \in \mathbb{Z}^n$. The set $M^{n,n}(\mathbb{Z})$ of $n \times n$ integer matrices acts on $\mathbb{Z}^n$ by left multiplication, inducing an action on monomials and hence on rational functions. It is not difficult to show:

Proposition 2.2. *For* $A \in M^{n,n}(\mathbb{Z})$ *with* $\det(A) \neq 0$,

$$m(A \cdot P) = m(P).$$

Corollary 2.3. *Let* $P(x_1, \ldots, x_n)$ *be a homogeneous polynomial. Then*

$$m\big(P(x_1, \ldots, x_n)\big) = m\big(P(1, x_2, \ldots, x_n)\big).$$

We will also make use of one additional fact, that Mahler measure is continuous as a function of coefficients (see [4]).

Theorem 2.4. *Fix positive integers* $d_1, \ldots, d_k$. *Let* $\mathbf{a}$ *represent a vector* $\{a(j_1, \ldots, j_k)\} \in \mathbb{C}^D$, *where* $0 \leq j_1 \leq d_1, \ldots, 0 \leq j_k \leq d_k$, *and* $D = (d_1 + 1) \cdots (d_k + 1)$. *Define a polynomial*

$$f_{\mathbf{a}} = \sum_{j_1=0}^{d_1} \cdots \sum_{j_k=0}^{d_k} a(j_1, \ldots, j_k)\, x_1^{j_1} \cdots x_k^{j_k}.$$

Then $m(f_{\mathbf{a}})$ *is continuous as a function of* $\mathbf{a}$ *on* $\mathbb{C}^D$.

2.2 Polylogarithms

A certain sequence of functions will play a central role in our calculations. For a positive integer k, the *kth polylogarithm function* is defined for $|z| < 1$ by

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k}.$$

(The name is explained by the observation that $\mathrm{Li}_1(z) = -\log(1-z)$.) We will refer to k as the *weight* of the polylogarithm. They may also be defined recursively for $k \geq 2$ by

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(z) \frac{dz}{z}.$$

These functions can be analytically continued beyond the unit disk, but they then become multivalued; the value of $\mathrm{Li}_k(z)$ depends on how the path of analytic continuation winds around 0 and 1.

One way to avoid this difficulty is to introduce a branch cut that prevents winding. The most common choice is the interval $(1, \infty) \subset \mathbb{R}$; the resulting continuation of $\mathrm{Li}_k(z)$ to $\mathbb{C} \setminus (1, \infty)$ is called the *principal branch*. Although the use of different branches in different contexts can simplify some expressions, the advantage of using a fixed branch is ease of communication and ease of calculation (most computer programs that implement polylogarithms use the principal branch). Henceforth, $\mathrm{Li}_k(z)$ will always refer to the principal branch.

As the coefficients for the power series for $\mathrm{Li}_k(z)$ are real, $\mathrm{Li}_k(z) \equiv \overline{\mathrm{Li}_k(\bar{z})}$ on the unit disc, and by continuation, on all of $\mathbb{C} \setminus [1, \infty)$. We may extend $\mathrm{Li}_k(r)$ to $r \in [1, \infty)$ by approaching r from either the upper or lower half plane; we will denote these values by $\mathrm{Li}_k^+(r)$ and $\mathrm{Li}_k^-(r)$, respectively. These values do not agree for $r > 1$, but since $\mathrm{Li}_k(\bar{z}) = \overline{\mathrm{Li}_k(z)}$, their real parts do agree. Therefore with $\mathrm{Re}[\mathrm{Li}_k(z)]$ we may drop the $\pm$ distinction, obtaining a function that is single-valued and continuous for all $z \in \mathbb{C}$, although not smooth as z crosses through $[1, \infty)$. Also, $\mathrm{Li}_k^+(1) = \mathrm{Li}_k^-(1)$, so $\mathrm{Li}_k(1)$ is well-defined.

A very different way to avoid the problem of multivaluedness is to use versions of the polylogarithm functions that are modified to make them single-valued. Several such modifications are in the literature (see e.g. [26]); the one most useful to us is due to Zagier [27]:

$$P_k(z) = \mathfrak{R}_k \left[\sum_{j=0}^{k} \frac{2^j B_j}{j!} \log^j |z| \, \mathrm{Li}_{k-j}(z) \right], \tag{2.3}$$

where $\mathfrak{R}_k[x]$ denotes $\mathrm{Re}[x]$ for k odd and $\mathrm{Im}[x]$ for k even, B_j is the j-th Bernoulli number, and $\mathrm{Li}_0(z) \equiv -1/2$ by convention. Although the summands are still discontinuous (or at least not smooth) along $(1, \infty)$, they precisely compensate for each other so that $P_k(z)$ extends to a single-valued function, C^∞ on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ and continuous on $\mathbb{P}^1(\mathbb{C})$. $P_2(z) = \mathrm{Im}\big[\mathrm{Li}_2(z)\big] + \log|z| \arg(1-z)$ is better known as $D(z)$, the Bloch-Wigner dilogarithm. (Here, as always, we take $\arg(x) \in (-\pi, \pi]$.)

One observation we will use is that for odd $k > 1$,

$$P_k(1) = \mathrm{Li}_k(1) = \zeta(k).$$

Polylogarithms satisfy a dizzying array of functional equations. If expressed in terms of $\mathrm{Li}_k(z)$, they tend to be messy; the equations often involve terms of different weights, and the domain of validity must be qualified. On the other hand, if expressed in terms of $P_k(z)$, both of these problems tend to disappear.

The simplest functional equations are:

$$P_k\left(\frac{1}{z}\right) = P_k(\bar{z}) = (-1)^{k-1} P_k(z). \tag{2.4}$$

One consequence of this is that for k even, $P_k(\bar{z}) = -P_k(z)$, so $P_k(z)$ vanishes on $\mathbb{R}$. (This is not the case for k odd.)

For $k, m \in \mathbb{N}$ and $\xi = e^{2\pi i/m}$, we have the "factorization theorem"

$$\frac{1}{m^{k-1}} P_k(z^m) = \sum_{j=0}^{m-1} P_k(\xi^j z). \tag{2.5}$$

This is a direct consequence of the well-known fact that

$$1 + \xi^n + \xi^{2n} + \ldots + \xi^{n(m-1)} = \begin{cases} m & \text{if } m \mid n, \\ 0 & \text{if } m \nmid n. \end{cases}$$

More impressive functional equations exist for particular weights. For instance, the dilogarithm satisfies the five-term relation, discovered independently by Spence and Abel:

$$D(x) + D(y) + D\Big(\frac{1-x}{1-xy}\Big) + D(1-xy) + D\Big(\frac{1-y}{1-xy}\Big) = 0. \tag{2.6}$$

Many simpler identities arise as special cases of this one, such as

$$D(1-x) = -D(x) \tag{2.7}$$

from setting $y = 0$, and

$$D(x) + D(-x) = \frac{1}{2} D(x^2) \tag{2.8}$$

from setting $y = x$. (The latter also follows from (2.5).)

It is worth noticing an additional structure in (2.6); if we define $z_0 = x$, $z_1 = y$, and for $k \geq 2$

$$z_k = \frac{1 - z_{k-2}}{1 - z_{k-2} z_{k-1}},$$

then the sequence of functions z_k is periodic in k, cyclically producing the five functions which appear as the arguments in (2.6). Also, if we instead define

$$z_k = \frac{1 - z_{k-1}}{z_{k-2}},$$

then this new sequence also has period five and gives us a different-looking but equivalent version of the five-term relation:

$$D(x) + D(y) + D\Big(\frac{1-y}{x}\Big) + D\Big(\frac{x+y-1}{xy}\Big) + D\Big(\frac{1-x}{y}\Big) = 0.$$

For the trilogarithm, we have the Spence-Kummer relation:

$$\begin{aligned} P_3\Big(\frac{x(1-y)^2}{y(1-x)^2}\Big) + P_3(xy) + P_3\Big(\frac{x}{y}\Big) - 2P_3\Big(\frac{x(1-y)}{y(1-x)}\Big) - 2P_3\Big(\frac{x(1-y)}{x-1}\Big) \\ - 2P_3\Big(\frac{y(1-x)}{y-1}\Big) - 2P_3\Big(\frac{1-y}{1-x}\Big) - 2P_3(x) - 2P_3(y) + 2P_3(1) = 0. \end{aligned} \tag{2.9}$$

Kummer also found identities involving polylogarithms of weights ≤ 4 and ≤ 5. In terms of Li_k, his identities are of mixed weight, but they have homogeneous versions in terms of P_k. For weight ≥ 6, no relations apart from (2.4) and (2.5) were known until very recently, when Gangl ([8], [9]) found identities for weights 6 and 7.

Chapter 3

Proof of a conjecture of Boyd

3.1 The conjecture

In early 2003, Boyd and Rodríguez Villegas were searching for polynomials $P(x, y, z)$ such that the intersection of the surfaces defined by $P = 0$ and $P^* = 0$ gave a curve E of genus 1. ($P^*(x, y, z)$, the *reciprocal polynomial* of $P(x, y, z)$, is the polynomial obtained by clearing denominators in $\overline{P}(x^{-1}, y^{-1}, z^{-1})$.) Based on ideas of Maillot, one might expect that, for such P, $m(P)$ would be a suitable multiple of $L(E, 3)$. While searching for such examples, they came across the following polynomial:

$$P(x, y, z) = 1 + x + (1 - x)(y + z) \tag{3.1}$$

For this polynomial, the associated curve E has genus 0, not 1. Boyd reasoned that for this degenerate case, $m(P)$ should be a rational multiple of $\zeta(3)/\pi^2$. Indeed, calculating $m(P)$ to high precision, he made the following conjecture, which we later proved.

Proposition 3.1.

$$m\big(1 + x + (1 - x)(y + z)\big) = \frac{28}{5\pi^2}\,\zeta(3).$$

The proof will proceed in three stages: reduction of the problem to integrating $\mathrm{Li}_2(z - z^{-1})\dfrac{dz}{z}$ over an arc, evaluation of the integral in terms of polylogarithms, and simplification of the expression using identities.

3.2 Reduction to a one-variable integral

My original approach to this part of the argument was to use a theorem of Maillot and Cassaigne [17]. Later, M. Lalín showed me how to shorten this step significantly, using the following result (which is essentially Theorem 12 in [12]). The proof given below follows the one in [12].

Theorem 3.2. *For $c \in \mathbb{C}$,*

$$m\big((1+x) + c(1-x)y\big) = \frac{2}{\pi}\,\mathrm{Im}\big[\mathrm{Li}_2(i|c|)\big].$$

Proof. First note that it is trivially true for $c = 0$. For $c \neq 0$, by Jensen's Formula,

$$\begin{aligned} m\big((1+x) + c(1-x)y\big) &= m(1+x) + m\Big(1 + c\frac{1-x}{1+x}y\Big) \\ &= \frac{1}{2\pi i}\int_{|x|=1} \log^{+}\left|c\frac{1-x}{1+x}\right|\frac{dx}{x}. \end{aligned} \tag{3.2}$$

If $x = e^{i\theta}$, then $\dfrac{1-x}{1+x} = -i\tan(\theta/2) \in i\,\mathbb{R}$. If we let $v = \Big(i|c|\dfrac{1-x}{1+x}\Big)^{-1}$, then $v \in \mathbb{R}$ and $x = -(1 - i|c|v)/(1 + i|c|v)$, so

$$\frac{dx}{x} = d\big[\log(1 - i|c|v) - \log(1 + i|c|v)\big] = 2i\,\mathrm{Im}\big[d\log(1 - i|c|v)\big].$$

So (3.2) becomes

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{+\infty}^{-\infty} \log^+|1/v| \cdot 2i\,\mathrm{Im}\big[d\log(1-i|c|v)\big] \\
&= \frac{1}{\pi}\,\mathrm{Im}\left[\int_{+1}^{-1} \log|1/v|\,d\log(1-i|c|v)\right] \\
&= \frac{2}{\pi}\,\mathrm{Im}\left[\int_0^1 \log(v)\,d\log(1-i|c|v)\right] \\
&= \frac{2}{\pi}\,\mathrm{Im}\left[\Big[\log(v)\log(1-i|c|v)\Big]_0^1 - \int_0^1 \log(1-i|c|v)\frac{dv}{v}\right] \\
&= \frac{2}{\pi}\,\mathrm{Im}\left[\int_0^1 d\,\mathrm{Li}_2(i|c|v)\right],
\end{aligned}$$

and the claim immediately follows. □

Let $C_\pm$ denote the upper and lower halves of the unit circle, oriented counterclockwise; it will also be convenient to let α denote C_-, negatively oriented. Using the above result, we may now reduce the problem to the integration of a holomorphic 1-form:

Proposition 3.3.

$$\pi^2\,m(P) = -2\,\mathrm{Re}\left[\int_\alpha \mathrm{Li}_2\big(z - z^{-1}\big)\frac{dz}{z}\right].$$

Proof. By propositions 2.1 and 2.2, the change of variables $y \to -yz^{-1}$, $z \to yz$ leaves $m(P)$ unchanged. So if we let $m_{x,y}$ denote Mahler measure, but integrating with respect to the variables x and y only (treating z as a constant), then

$$\begin{aligned}
\pi^2\,m(P) &= \pi^2\,m\big(1 + x + (z - z^{-1})(1-x)y\big) \\
&= \frac{\pi^2}{2\pi i}\int_{|z|=1} m_{x,y}\big(1 + x + (z - z^{-1})(1-x)y\big)\frac{dz}{z}.
\end{aligned}$$

By theorem 3.2, this becomes

$$-i\int_{|z|=1} \mathrm{Im}\Big[\mathrm{Li}_2\big(i|z-z^{-1}|\big)\Big]\,\frac{dz}{z}.$$

Since $-i\,dz/z = d\arg(z)$ is a real form, the above is equal to

$$\int_{|z|=1} \mathrm{Im}\Big[\mathrm{Li}_2\big(i|z-z^{-1}|\big)\,\Big(-i\frac{dz}{z}\Big)\Big] \;=\; -\,\mathrm{Re}\Big[\int_{|z|=1} \mathrm{Li}_2\big(i|z-z^{-1}|\big)\,\frac{dz}{z}\Big]$$

For $|z|=1$, $z-z^{-1}=z-\bar{z}\in i\,\mathbb{R}$, so $i|z-z^{-1}| = \mathrm{sign}(\mathrm{Im}[z])\cdot(z-z^{-1})$. So our expression becomes

$$-\,\mathrm{Re}\Big[\int_{C_+} \mathrm{Li}_2\big(z-z^{-1}\big)\frac{dz}{z} \;+\; \int_{C_-} \mathrm{Li}_2\big(-z+z^{-1}\big)\frac{dz}{z}\Big].$$

In the first integral, we perform the change of variables $z\to\bar{z}=z^{-1}$; in the second, we simply observe that $z=\bar{z}^{-1}$:

$$\begin{aligned}
&-\,\mathrm{Re}\Big[\int_{\alpha} \mathrm{Li}_2\big(\overline{z-z^{-1}}\big)\Big(-\frac{dz}{z}\Big) \;+\; \int_{C_-} \mathrm{Li}_2\big(\overline{-z^{-1}+z}\big)\frac{dz}{z}\Big]\\
=\;&-\,\mathrm{Re}\Big[\int_{\alpha} \overline{\mathrm{Li}_2\big(z-z^{-1}\big)\frac{dz}{z}} \;+\; \int_{\alpha} \overline{\mathrm{Li}_2\big(z-z^{-1}\big)\frac{dz}{z}}\Big],
\end{aligned}$$

and the claim follows. □

3.3 Evaluation of the integral

As aesthetically pleasing as the expression in proposition 3.3 may be, we do not know how to evaluate it directly. Instead, in this section we will rephrase this integral in terms of integrals of the form

$$\int \log(f)\log(g)\frac{dz}{z} \qquad \text{and} \qquad \int \log(f)\frac{dz}{z},$$

where f and g are linear, of the form z or $1-cz$. We are able to evaluate such integrals using polylogarithms.

Performing integration by parts on the right side of proposition 3.3, we obtain

$$\begin{aligned}\pi^2\, m(P) = & -2\operatorname{Re}\Bigg[\Big[\operatorname{Li}_2\big(z-z^{-1}\big)\log(z)\Big]_{\partial\alpha} \\ & \qquad + \int_\alpha \log(z)\log\big(1-z+z^{-1}\big)\left(\frac{1+z^{-2}}{z-z^{-1}}\right)dz\Bigg] \\ = & -2\operatorname{Re}\Bigg[\int_\alpha \log(z)\log\big(1-z+z^{-1}\big)\left(\frac{1+z^{-2}}{z-z^{-1}}\right)dz\Bigg]. \qquad (3.3)\end{aligned}$$

Expanding in partial fractions,

$$\frac{1+z^{-2}}{z-z^{-1}} = \frac{1}{z+1} - \frac{1}{z} + \frac{1}{z-1}.$$

Under $z \to -z^{-1}$, the expression $1-z+z^{-1}$ is invariant, and the orientation on α is reversed. Hence,

$$\begin{aligned}&\int_\alpha \log(z)\log(1-z+z^{-1})\frac{dz}{z+1} \\ &\qquad = -\int_\alpha \log(-z^{-1})\log(1-z+z^{-1})\frac{d(-z^{-1})}{(-z^{-1}+1)} \\ &\qquad = \int_\alpha \log(-z)\log(1-z+z^{-1})\frac{dz}{z(z-1)} \\ &\qquad = \int_\alpha \big(\log(z)+\pi i\big)\log(1-z+z^{-1})\left(\frac{1}{z-1}-\frac{1}{z}\right)dz.\end{aligned}$$

So (3.3) equals

$$\begin{aligned}&-4\operatorname{Re}\Bigg[\int_\alpha \log(z)\log(1-z+z^{-1})\left(\frac{1}{z-1}-\frac{1}{z}\right)dz\Bigg] \\ &\qquad + 2\pi\operatorname{Im}\Bigg[\int_\alpha \log(1-z+z^{-1})\left(\frac{1}{z-1}-\frac{1}{z}\right)dz\Bigg]. \qquad (3.4)\end{aligned}$$

Let $\mathfrak{H}$ denote the upper half plane, and $\varphi = \dfrac{1+\sqrt{5}}{2}$. For $z \in -\mathfrak{H}$,

$$\log(1 - z + z^{-1}) \;=\; -\log(z) + \log(1 - \varphi^{-1} z) + \log(1 + \varphi z),$$

and for $z \in \mathfrak{H}$,

$$\begin{aligned} \log\bigl(1 - (1-z) + (1-z)^{-1}\bigr) &= \log\left(\frac{z}{1-z}\bigl(1 - z + z^{-1}\bigr)\right) \\ &= -\log(1-z) + \log(1 - \varphi^{-1} z) + \log(1 + \varphi z). \end{aligned}$$

For functions $f(z)$, $g(z)$ and a path γ, we use the notation

$$\begin{aligned} \mathrm{I}_\gamma(f,g) &= \int_\gamma \log(f)\,\log(g)\,\frac{dz}{z} \\ \mathrm{J}_\gamma(f,g) &= \int_\gamma \log(f)\,\frac{dz}{z} \end{aligned}$$

If we multiply out (3.4), perform the substitution $z \to -z^{-1}$ in the terms containing $1/(z-1)$, and expand the logarithms as above, we obtain

$$\begin{aligned} -4\,\mathrm{Re}\Biggl[& \mathrm{I}_\alpha(z,z) \;-\; \mathrm{I}_\alpha(z, 1-\varphi^{-1}z) \;-\; \mathrm{I}_\alpha(z, 1+\varphi z) \\ & - \mathrm{I}_{1-\alpha}(1-z, 1-z) \;+\; \mathrm{I}_{1-\alpha}(1-z, 1-\varphi^{-1}z) \;+\; \mathrm{I}_{1-\alpha}(1-z, 1+\varphi z)\Biggr] \\ +\,2\pi\,\mathrm{Im}\Biggl[& \mathrm{J}_\alpha(z) \;-\; \mathrm{J}_\alpha(1-\varphi^{-1}z) \;-\; \mathrm{J}_\alpha(1+\varphi z) \\ & - \mathrm{J}_{1-\alpha}(1-z) \;+\; \mathrm{J}_{1-\alpha}(1-\varphi^{-1}z) \;+\; \mathrm{J}_{1-\alpha}(1+\varphi z)\Biggr]. \end{aligned} \tag{3.5}$$

The J_α and $\mathrm{J}_{1-\alpha}$ terms are easy to evaluate:

$$\mathrm{J}_\alpha(z) \;=\; \int_\alpha \log(z)\frac{dz}{z} \;=\; \left[\frac{1}{2}\log^2(z)\right]_{\partial\alpha} \;=\; -\frac{\pi^2}{2}.$$

And for any $c \neq 0$ and path γ,

$$\begin{aligned}
\mathrm{J}_\gamma(1-cz) &= \int_\gamma \log(1-cz)\frac{dz}{z} \\
&= \int_{c^{-1}\gamma} \log(1-z)\frac{dz}{z} \\
&= \left[-\operatorname{Li}_2(z)\right]_{\partial(c^{-1}\gamma)}.
\end{aligned}$$

Hence:

$$\begin{aligned}
\mathrm{J}_\alpha(1-\varphi^{-1}z) &= \operatorname{Li}_2(\varphi^{-1}) - \operatorname{Li}_2(-\varphi^{-1}) \\
\mathrm{J}_\alpha(1+\varphi z) &= \operatorname{Li}_2(-\varphi) - \operatorname{Li}_2^+(\varphi) \\
\mathrm{J}_{1-\alpha}(1-z) &= -\operatorname{Li}_2^+(2) \\
\mathrm{J}_{1-\alpha}(1-\varphi^{-1}z) &= -\operatorname{Li}_2^+(2\varphi^{-1}) \\
\mathrm{J}_{1-\alpha}(1+\varphi z) &= -\operatorname{Li}_2(-2\varphi)
\end{aligned}$$

The Bloch-Wigner dilogarithm $D(z) = \operatorname{Im}\big[\operatorname{Li}_2(z)\big] + \log|z|\arg(1-z)$ vanishes on $\mathbb{R}$. Consequently, for real r,

$$\operatorname{Im}\big[\operatorname{Li}_2^\pm(r)\big] = \begin{cases} 0, & \text{if } r \leq 1, \\ \pm\pi\log r & \text{if } r > 1. \end{cases} \tag{3.6}$$

So we have

$$\begin{aligned}
2\pi\operatorname{Im}\Bigg[&\mathrm{J}_\alpha(z) - \mathrm{J}_\alpha(1-\varphi^{-1}z) - \mathrm{J}_\alpha(1+\varphi z) \\
&- \mathrm{J}_{1-\alpha}(1-z) + \mathrm{J}_{1-\alpha}(1-\varphi^{-1}z) + \mathrm{J}_{1-\alpha}(1+\varphi z)\Bigg] \\
&= 2\pi\operatorname{Im}\big[\operatorname{Li}_2^+(\varphi) + \operatorname{Li}_2^+(2) - \operatorname{Li}_2^+(2\varphi^{-1})\big] \\
&= 2\pi\big(\pi\log(\varphi) + \pi\log(2) - \pi\log(2\varphi^{-1})\big) \\
&= 4\pi^2\log\varphi.
\end{aligned} \tag{3.7}$$

The I_α terms are also manageable:

$$\begin{aligned}
\operatorname{Re}\left[\mathrm{I}_\alpha(z,z)\right] &= \operatorname{Re}\left[\int_\alpha \log^2(z)\frac{dz}{z}\right] \\
&= \operatorname{Re}\Big[\big[\tfrac{1}{3}\log^3(z)\big]_{\partial\,\alpha}\Big] \;=\; \frac{1}{3}\operatorname{Re}\big[(\pi i)^3\big] \;=\; 0.
\end{aligned} \tag{3.8}$$

In the integral defining $\mathrm{I}_\alpha(z, 1+\varphi z)$, we perform the substitution $z \to -z^{-1}$:

$$\begin{aligned}
\mathrm{I}_\alpha(z,1+\varphi z) &= -\int_\alpha \log(-z^{-1})\log(1-\varphi z^{-1})\left(-\frac{dz}{z}\right) \\
&= -\int_\alpha \big(\log z + \pi i\big)\big(\log(1-\varphi^{-1}z) - \log z + \log\varphi - \pi i\big)\frac{dz}{z} \\
&= -\,\mathrm{I}_\alpha(z,1-\varphi^{-1}z) \;+\; \mathrm{I}_\alpha(z,z) \;+\; (-\log\varphi + 2\pi i)\,\mathrm{J}_\alpha(z) \\
&\qquad -\pi i\,\mathrm{J}_\alpha(1-\varphi^{-1}z) - \pi i(\log\varphi - \pi i)\int_\alpha \frac{dz}{z} \\
&= -\,\mathrm{I}_\alpha(z,1-\varphi^{-1}z) \;+\; \mathrm{I}_\alpha(z,z) \;-\; \frac{\pi^2}{2}\big(-\log\varphi + 2\pi i\big) \\
&\qquad -\pi i\big(\mathrm{Li}_2(\varphi^{-1}) - \mathrm{Li}_2(-\varphi^{-1})\big) - \pi i(\log\varphi - \pi i)(-\pi i),
\end{aligned}$$

hence

$$\operatorname{Re}\Big[\mathrm{I}_\alpha(z,1-\varphi^{-1}z) \;+\; \mathrm{I}_\alpha(z,1+\varphi z)\Big] \;=\; -\frac{\pi^2}{2}\log\varphi. \tag{3.9}$$

It remains to evaluate the three $\mathrm{I}_{1-\alpha}$ terms. For these, we make use of the following astonishing identity, found as equation (8.111) in Lewin's book [15]:

$$\begin{aligned}
&\int_0^x \log(1-z)\log(1-cz)\frac{dz}{z} \\
&= \mathrm{Li}_3\Big(\frac{1-cx}{1-x}\Big) \;+\; \mathrm{Li}_3(1/c) \;+\; \mathrm{Li}_3(1) \;-\; \mathrm{Li}_3(1-cx) \\
&\quad -\; \mathrm{Li}_3(1-x) \;-\; \mathrm{Li}_3\Big(\frac{1-cx}{c(1-x)}\Big) \;+\; \log(1-cx)\Big[\mathrm{Li}_2(1/c) - \mathrm{Li}_2(x)\Big] \\
&\quad +\; \log(1-x)\Big[\mathrm{Li}_2(1-cx) - \mathrm{Li}_2(1/c) + \mathrm{Li}_2(1)\Big] \;+\; \frac{1}{2}\log(c)\log^2(1-x).
\end{aligned} \tag{3.10}$$

A remark on the interpretation of this identity is necessary. We will only be interested here in values of $c \in \mathbb{R}^\times$ and $x \in \mathbb{C} \setminus \mathbb{R}$ (or more accurately, x approaching $\mathbb{R}$ from $\pm\mathfrak{H}$). If $c < 1$, then the quantities $\log(c)$, $\mathrm{Li}_2(1/c)$ and $\mathrm{Li}_3(1/c)$ are ambiguous. For certain choices of them, we can make the identity true on $\mathfrak{H}$ or $-\mathfrak{H}$, though not both at once. Indeed, to make the identity valid for $x \in \pm\mathfrak{H}$, we choose:

$$\begin{aligned} \mathrm{Li}_3(1/c) &= \mathrm{Li}_3^{\pm}(1/c) \\ \mathrm{Li}_2(1/c) &= \mathrm{Li}_2^{\pm}(1/c) \\ \log(c) &= \lim_{z\to c}{}^{\mp} \log(z), \end{aligned}$$

where the symbols $\lim^+_{z\to c}$ and $\lim^-_{z\to c}$ mean that the limit is taken as z approaches c from the upper and lower half planes, respectively.

A fuller discussion of (3.10), including how to extend it to $c \notin \mathbb{R}$, will be given in section 4.2.

From the version of (3.10) valid on $+\mathfrak{H}$, with $c = 1$, we obtain

$$\begin{aligned} &\mathrm{Re}\big[\mathrm{I}_{1-\alpha}(1-z, 1-z)\big] \\ &\qquad = \mathrm{Re}\Big[2\,\mathrm{Li}_3(1) - 2\,\mathrm{Li}_3(1) - \pi i\big(-\mathrm{Li}_2^{+}(2) + \mathrm{Li}_2(-1) + \mathrm{Li}_2(1)\big)\Big] \\ &\qquad = \frac{7}{2}\zeta(3) - \pi^2 \log 2, \end{aligned} \tag{3.11}$$

using (3.6) along with the facts that $\mathrm{Li}_3(1) = \zeta(3)$ and $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$ (which follows from (2.5)).

From $c=\varphi^{-1}$,

$$
\begin{aligned}
&\operatorname{Re}\bigl[\mathrm{I}_{1-\alpha}(1-z,1-\varphi^{-1}z)\bigr]\\
&\quad= \operatorname{Re}\Bigl[\mathrm{Li}_3(2\varphi^{-1}-1) + \mathrm{Li}_3(\varphi) + \mathrm{Li}_3(1) - \mathrm{Li}_3(1-2\varphi^{-1}) - \mathrm{Li}_3(-1)\\
&\qquad\qquad - \mathrm{Li}_3(2-\varphi) + \bigl(\log(2\varphi^{-1}-1)-\pi i\bigr)\bigl(\mathrm{Li}_2^{+}(\varphi)-\mathrm{Li}_2^{+}(2)\bigr)\\
&\qquad\qquad + (-\pi i)\bigl(\mathrm{Li}_2(1-2\varphi^{-1}) - \mathrm{Li}_2^{+}(\varphi)+\pi^2/6\bigr) + \frac{1}{2}\log(\varphi^{-1})(-\pi i)^2\Bigr]\\
&\quad= \operatorname{Re}\Bigl[\mathrm{Li}_3(\varphi^{-3}) - \mathrm{Li}_3(-\varphi^{-3}) + \mathrm{Li}_3(\varphi) - \mathrm{Li}_3(\varphi^{-2}) + \frac{7}{4}\zeta(3)\Bigr]\\
&\qquad - 3\log(\varphi)\operatorname{Re}\bigl[\mathrm{Li}_2(\varphi) - \mathrm{Li}_2(2)\bigr] - \pi^2\log(2) + \frac{\pi^2}{2}\log(\varphi),
\end{aligned}
\tag{3.12}
$$

using that $2\varphi^{-1}-1=\varphi^{-3}$ and $2-\varphi=\varphi^{-2}$.

From $c=-\varphi$,

$$
\begin{aligned}
&\operatorname{Re}\bigl[\mathrm{I}_{1-\alpha}(1-z,1+\varphi z)\bigr]\\
&\quad= \operatorname{Re}\Bigl[\mathrm{Li}_3(-2\varphi-1) + \mathrm{Li}_3(-\varphi^{-1}) + \mathrm{Li}_3(1) - \mathrm{Li}_3(2\varphi+1) - \mathrm{Li}_3(-1)\\
&\qquad\qquad - \mathrm{Li}_3(2+\varphi^{-1}) + \log(2\varphi+1)\bigl(\mathrm{Li}_2(-\varphi^{-1})-\mathrm{Li}_2^{+}(2)\bigr)\\
&\qquad\qquad - \pi i\Bigl(\mathrm{Li}_2^{+}(2\varphi+1) - \mathrm{Li}_2(-\varphi^{-1}) + \frac{\pi^2}{6}\Bigr) + \frac{1}{2}\bigl(\log\varphi-\pi i\bigr)(-\pi i)^2\\
&\quad= \operatorname{Re}\Bigl[\mathrm{Li}_3(-2\varphi-1) - \mathrm{Li}_3(2\varphi+1) + \mathrm{Li}_3(-\varphi^{-1}) - \mathrm{Li}_3(\varphi^2) + \frac{7}{4}\zeta(3)\Bigr]\\
&\qquad + 3\log(\varphi)\operatorname{Re}\bigl[\mathrm{Li}_2(-\varphi^{-1}) - \mathrm{Li}_2(2)\bigr] + \frac{5\pi^2}{2}\log(\varphi),
\end{aligned}
\tag{3.13}
$$

using that $2\varphi+1=\varphi^3$ and $2+\varphi^{-1}=\varphi^2$.

Combining (3.3)–(3.13), we have

$$
\begin{aligned}
\pi^2\, m(P) = -4\operatorname{Re}\Bigl[&\mathrm{Li}_3(-\varphi^3) - \mathrm{Li}_3(\varphi^3) + \mathrm{Li}_3(\varphi^{-3}) - \mathrm{Li}_3(-\varphi^{-3})\\
&- \mathrm{Li}_3(\varphi^2) - \mathrm{Li}_3(\varphi^{-2}) + \mathrm{Li}_3(\varphi) + \mathrm{Li}_3(-\varphi^{-1})\Bigr]\\
&+ 12\log(\varphi)\operatorname{Re}\bigl[\mathrm{Li}_2(\varphi) - \mathrm{Li}_2(-\varphi^{-1})\bigr] - 10\pi^2\log(\varphi).
\end{aligned}
\tag{3.14}
$$

We have now evaluated $\pi^2 m(P)$ using essentially only polylogarithms of algebraic numbers. But as of this point, it is unclear how or if this eleven term combination of logarithms, dilogarithms and trilogarithms should simplify to a rational multiple of $\mathrm{Li}_3(1) = \zeta(3)$. The demonstration of this is the purpose of the next section.

3.4 Simplification of the expression

We will now need to make use of a number of polylogarithm identities. These will for the most part be quoted from [15] without proof, although they may all be deduced from (2.4)–(2.9), as is demonstrated for the example immediately below.

Lewin's identity (1.11) says that

$$\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) \;=\; \frac{\pi^2}{6} - \log(z)\log(1-z).$$

(This may be deduced from (2.7) by observing that $0 = D(z) + D(1-z) = \mathrm{Im}\big[\mathrm{Li}_2(z) + \mathrm{Li}_2(1-z) + \log(z)\log(1-z)\big]$, and that holomorphic functions mapping into $\mathbb{R}$ must be constant; the constant is found by letting $z \to 1$.) Although only valid on $\mathbb{R}$ for $0 < z < 1$, it may be analytically continued to $\pm\mathfrak{H}$. Approaching $z = \varphi$ from $\mathfrak{H}$ and using that $1 - \varphi = -\varphi^{-1}$,

$$\begin{aligned} \mathrm{Re}\big[\mathrm{Li}_2(\varphi)\big] &= \mathrm{Re}\Big[-\mathrm{Li}_2(-\varphi^{-1}) \;+\; \frac{\pi^2}{6} \;-\; \log(\varphi)\big(\log(\varphi^{-1}) - \pi i\big)\Big] \\ &= -\,\mathrm{Li}_2(-\varphi^{-1}) \;+\; \frac{\pi^2}{6} \;+\; \log^2(\varphi). \end{aligned}$$

According to Lewin's (1.21),

$$\mathrm{Li}_2(-\varphi^{-1}) \;=\; -\frac{\pi^2}{15} + \frac{1}{2}\log^2(\varphi),$$

so the last line of (3.14) reduces to

$$12\log(\varphi)\operatorname{Re}\bigl[\operatorname{Li}_2(\varphi)-\operatorname{Li}_2(-\varphi^{-1})\bigr]\;-\;10\pi^2\log(\varphi)=\;-\frac{32}{5}\pi^2\log\varphi. \tag{3.15}$$

This eliminates all dilogarithms from our expression.

We will also need Lewin's identities (6.4), (6.6), (6.10) and (6.33):

$$\tfrac{1}{4}\operatorname{Li}_3(x^2)\;=\;\operatorname{Li}_3(x)+\operatorname{Li}_3(-x) \tag{3.16}$$

$$\operatorname{Li}_3(-x)-\operatorname{Li}_3(-1/x)\;=\;-\frac{\pi^2}{6}\log x-\tfrac{1}{6}\log^3 x \tag{3.17}$$

$$\begin{aligned}&\operatorname{Li}_3\Bigl(\frac{-x}{1-x}\Bigr)+\operatorname{Li}_3(1-x)+\operatorname{Li}_3(x)\\&\quad=\operatorname{Li}_3(1)+\frac{\pi^2}{6}\log(1-x)-\tfrac{1}{2}\log(x)\log^2(1-x)+\tfrac{1}{6}\log^3(1-x)\end{aligned} \tag{3.18}$$

$$\begin{aligned}\operatorname{Li}_3\Bigl(\frac{1-x}{1+x}\Bigr)-\operatorname{Li}_3\Bigl(\frac{x-1}{x+1}\Bigr)&=2\operatorname{Li}_3(1-x)+2\operatorname{Li}_3\Bigl(\frac{1}{x+1}\Bigr)-\frac{1}{2}\operatorname{Li}_3(1-x^2)\\&\quad-\frac{7}{4}\operatorname{Li}_3(1)+\frac{\pi^2}{6}\log(x+1)-\frac{1}{3}\log^3(x+1)\end{aligned} \tag{3.19}$$

Each of these identities is valid for those $x\in\mathbb{R}$ such that the arguments of all logarithms are positive and the arguments of all polylogarithms are ≤ 1. But note that most of the arguments above have the form $(ax+b)/(cx+d)$ for $a,b,c,d\in\mathbb{R}$. Such linear fractional transformations take real values only if x is real; consequently, $\operatorname{Li}_3\bigl(\frac{ax+b}{cx+d}\bigr)$ and $\log\bigl(\frac{ax+b}{cx+d}\bigr)$ extend analytically to $\pm\mathfrak{H}$. Also, the principal branch of $\operatorname{Li}_3(x^2)$ is holomorphic on $\mathfrak{H}$. Hence, (3.16), (3.17) and (3.18) extend to $\mathfrak{H}$.

We will also need to extend (3.19) so that we may let x approach $-\varphi$ from $\mathfrak{H}$. All terms in this identity extend to $\pm\mathfrak{H}$ except $\mathrm{Li}_3(1-x^2)$. This function is holomorphic on the left and right half planes, but as x crosses the imaginary axis, then $1-x^2 \in [1,\infty)$, so the analytic continuation will not be the principal branch. Indeed, as $\mathrm{Li}_3(z)$ is analytically continued along a path from $-\mathfrak{H}$, passing through $(1,\infty)$, to some $z_0 \in \mathfrak{H}$, the value obtained is $\mathrm{Li}_3(z_0) - \pi i \log^2 z_0$, where the principal branches are again used (see, e.g., [1]). Hence, as we analytically continue $\mathrm{Li}_3(1-x^2)$ along a path through $\mathfrak{H}$ from the first quadrant to $-\varphi$, $z = 1-x^2$ moves along the type of path described above, so we reach the value

$$\mathrm{Li}_3(1-\varphi^2) - \pi i\big(\log^+(1-\varphi^2)\big)^2 \;=\; \mathrm{Li}_3(-\varphi) + 2\pi^2\log\varphi - \pi i\big(\log^2\varphi - \pi^2\big).$$

Now, taking (3.19) with $x \to -\varphi$, (3.19) with $x=\varphi$, (3.18) with $x \to \varphi$, (3.18) with $x \to -\varphi^{-1}$, (3.16) with $x \to \varphi$ and (3.17) with $x = \varphi$ (all limits are as x approaches from $\mathfrak{H}$), we obtain six equations, encoded in matrix form as

$$M\,v \;=\; \mathbf{0} \tag{3.20}$$

where

$$M = \begin{bmatrix} 7/4 & 1 & -1 & 0 & 0 & -2 & 0 & 0 & 0 & -3/2 & -1/3 & 13/6 \\ 7/4 & 0 & 0 & -1 & 1 & 0 & -2 & 0 & -2 & 1/2 & 8/3 & -1/3 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2/3 & -5/6 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & -2/3 & -1/6 \\ 0 & 0 & 0 & 0 & 0 & 1/4 & 0 & -1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1/6 & 1/6 \end{bmatrix}$$

and

$$v = \begin{bmatrix} \zeta(3) \\ \mathrm{Re}\big[\mathrm{Li}_3(-\varphi^3)\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(\varphi^3)\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(\varphi^{-3})\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(-\varphi^{-3})\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(\varphi^2)\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(\varphi^{-2})\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(\varphi)\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(-\varphi^{-1})\big] \\ \mathrm{Re}\big[\mathrm{Li}_3(-\varphi)\big] \\ \log^3\varphi \\ \pi^2\log\varphi \end{bmatrix}$$

Multiplying (3.20) on the left by the row vector

$$\big[1, \ \ -1, \ \ 8/5, \ \ -3, \ \ -12/5, \ \ -2/5\big]$$

shows that

$$-\frac{7}{5}\zeta(3) - \frac{8}{5}\pi^2\log\varphi = \mathrm{Re}\Big[\mathrm{Li}_3(-\varphi^3) \, - \, \mathrm{Li}_3(\varphi^3) \, + \, \mathrm{Li}_3(\varphi^{-3}) \, - \, \mathrm{Li}_3(-\varphi^{-3}) \\ - \mathrm{Li}_3(\varphi^2) \, - \, \mathrm{Li}_3(\varphi^{-2}) \, + \, \mathrm{Li}_3(\varphi) \, + \, \mathrm{Li}_3(-\varphi^{-1})\Big].$$

Combining this with (3.14) and (3.15), we finally have

$$\begin{aligned} \pi^2\, m(P) &= \, -4\Big(-\frac{7}{5}\,\zeta(3) - \frac{8}{5}\,\pi^2\log\varphi\Big) - \frac{32}{5}\,\pi^2\log\varphi \\ &= \frac{28}{5}\,\zeta(3). \end{aligned}$$

3.5 Remarks

As seen in (3.20), there are a large number of linear relations among the numbers $\mathrm{Re}\big[\mathrm{Li}_3(\pm\varphi^{\pm k})\big]$, for $1 \le k \le 3$ (and there are others which were omitted). As such, it is maybe not too surprising that one could reduce all the trilogarithms in (3.14) down to just $\zeta(3)$, along with terms of lower weight. But the fact that the lower-weight terms completely evaporate is something of a miracle and begs for an explanation.

One idea in this direction is to avoid the principal branches and instead to use the modified polylogarithms $P_k(z)$, as their functional equations have no lower-weight terms. Indeed, if we take equation (3.14), drop all the polylogarithms of weight less than 3, and replace $\mathrm{Re}\big[\mathrm{Li}_3(z)\big]$ with $P_3(z)$, the equation appears to still hold:

$$\begin{aligned}\frac{28}{5}\zeta(3) \overset{?}{=} -4\Big(& P_3(-\varphi^3) - P_3(\varphi^3) + P_3(\varphi^{-3}) - P_3(-\varphi^{-3}) \\ & - P_3(\varphi^2) - P_3(\varphi^{-2}) + P_3(\varphi) + P_3(-\varphi^{-1})\Big) \end{aligned} \tag{3.21}$$

This equation has been verified numerically to over twenty decimal places using the program PARI, but we have not proven it. We expect that a proof could be found by simply mimicking the arguments of the previous section. Of course, this does not fully resolve the miracle, as it does not explain how one would get from $\pi^2 m(P)$ to (3.21). We discuss these ideas further in chapter 5.

Chapter 4

Extensions to families of polynomials

4.1 Results

The methods of the previous chapter can be extended to calculate the Mahler measure for several large families of polynomials. These examples involve polynomials in three variables and give evaluations in terms of polylogarithms of weight ≤ 3. Further, the arguments of the polylogarithms will be algebraic numbers if the coefficients of the polynomials are algebraic.

We will give examples of two such families. To make the statements explicit, we use the following notation. Fix an embedding of $\overline{\mathbb{Q}}$ in $\mathbb{C}$ and define

$$S_1 = \left\{\log(\alpha)\log(\beta)\log(\gamma) \ : \ \alpha,\beta,\gamma \in \overline{\mathbb{Q}}\right\}$$

$$S_2 = \left\{\log(\alpha)\,\mathrm{Li}_2(\beta) \ : \ \alpha,\beta \in \overline{\mathbb{Q}}\right\}$$

$$S_3 = \left\{\mathrm{Li}_3(\alpha) \ : \ \alpha \in \overline{\mathbb{Q}}\right\}$$

$$T_1 = \left\{\log(\alpha)\log(\beta) \ : \ \alpha,\beta \in \overline{\mathbb{Q}}\right\}$$

$$T_2 = \left\{\mathrm{Li}_2(\alpha) \ : \ \alpha \in \overline{\mathbb{Q}}\right\}$$

For $\alpha \in \mathbb{R}$, we wish to include both $\mathrm{Li}_k^+(\alpha)$ and $\mathrm{Li}_k^-(\alpha)$, but this is automatic, since it may be shown that for all $k \geq 1$ and $\alpha \in [1, \infty)$,

$$\mathrm{Li}_k^+(\alpha) = \mathrm{Li}_k^-(\alpha) + 2\pi i \, \frac{\log^{k-1}(\alpha)}{(k-1)!}.$$

We define $\mathcal{S}$ and $\mathcal{T} \subset \mathbb{C}$ to be the $\mathbb{Q}$-vector spaces generated by the elements of $S_1 \cup S_2 \cup S_3$ and $T_1 \cup T_2$, respectively.

Also, recall that the *reciprocal* of a polynomial $P(z)$ is defined to be

$$P^\star(z) = z^{\deg P} \cdot \overline{P}\big(z^{-1}\big),$$

and P is said to be *reciprocal* if $P = P^\star$. Then we have

Theorem 4.1. *For $R(z) \in \overline{\mathbb{Q}}(z)$ equal to the product of a reciprocal polynomial of even degree with the square of a rational function,*

$$m\big(1 + x + R(z)(1-x)y\big) \; \in \; \frac{1}{\pi^2}\,\mathcal{S}.$$

Theorem 4.2. *For any rational function $R(z) \in \overline{\mathbb{Q}}(z)$,*

$$m\big(x + 1 - R(z)(y+1)\big) \; \in \; \begin{cases} \dfrac{1}{\pi i}\,\mathcal{T} & \textit{if } R(z) \textit{ maps the unit circle into } \mathbb{R}, \\ \dfrac{1}{\pi^2}\,\mathcal{S} & \textit{otherwise.} \end{cases}$$

Proofs of these theorems will be provided in sections 4.3 and 4.4.

Observe that if we assign weights of 1 to each Mahler measure value and to π, and of k to each value of $\mathrm{Li}_k(z)$, then the above evaluations are all homogeneous. Further, for the cases involving $\mathcal{S}$, each monomial (after clearing the powers of π out of the denominator) has weight three—the number of variables inside the Mahler measure. (The case where $\mathcal{T}$ appears may be viewed as degenerate.) This homogeneity was previously observed by Lalín

[12] for some evaluations of her own, and it appears to be a property of all Mahler measure evaluations of polynomials with algebraic coefficients in terms of polylogarithms of algebraic numbers.

For both 4.1 and 4.2, the evaluations can be made completely explicit for any specific choice of $R(z)$. Unfortunately, the evaluations tend to be enormous, having far more than the eleven terms we saw in (3.14). And while the use of identities can simplify these polylogarithm expressions somewhat, in general we see no reason to expect them to simplify down to a single term, as happened in the previous example.

We would like to give formulas for the Mahler measures described above, but these would be exceedingly messy. Instead we will prove the theorems by describing how to rewrite the Mahler measures in terms of certain logarithmic integrals (discussed in the next section), which themselves evaluate in terms of polylogarithms. As a compromise, in section 4.5 we will more fully work out the evaluation for one particular subfamily.

4.2 Integrating logarithmic forms

Recall from section 3.3 that the strategy for evaluating $m(P)$ was to rephrase the integral as a sum of integrals of the form

$$\int \log(f)\log(g)\frac{dz}{z} \qquad \text{and} \qquad \int \log(f)\frac{dz}{z}, \tag{4.1}$$

where f and g are polynomials. Complex logarithms satisfy

$$\log(ab) = \log(a) + \log(b) + 2\pi k$$

with $k \in \mathbb{Z}$ depending on a and b, but remaining constant as long as a, b and ab do not cross $(-\infty, 0]$. So, factoring f and g and expanding the logarithms, we see that the integrals in (4.1) can be expressed as linear combinations of similar integrals, as well as $\int dz/z$, but now with f and g of the form z or $1 - cz$. Most of these building blocks are easy to deal with:

$$
\begin{aligned}
\log(z)\frac{dz}{z} &= d\Big[\frac{1}{2}\log^2(z)\Big] \\
\log^2(z)\frac{dz}{z} &= d\Big[\frac{1}{3}\log^3(z)\Big] \\
\log(1-cz)\frac{dz}{z} &= d\Big[-\operatorname{Li}_2(cz)\Big] \\
\log(z)\log(1-cz)\frac{dz}{z} &= d\Big[\operatorname{Li}_3(cz) - \log(z)\operatorname{Li}_2(cz)\Big]
\end{aligned}
\tag{4.2}
$$

$$\log^2(1-cz)\frac{dz}{z} = d\Big[-2\operatorname{Li}_3(1-cz)+2\log(1-cz)\operatorname{Li}_2(1-cz)+\log(cz)\log^2(1-cz)\Big]$$

The remaining case was previously discussed in equation (3.10). We repeat that result here (slightly modified) for convenience:

$$
\begin{aligned}
&\log(1-z)\log(1-cz)\frac{dz}{z} \\
&= d\Bigg[\operatorname{Li}_3\Big(\frac{1-cz}{1-z}\Big) - \operatorname{Li}_3(1-cz) - \operatorname{Li}_3(1-z) - \operatorname{Li}_3\Big(\frac{1-cz}{c(1-z)}\Big) \\
&\qquad + \log(1-z)\Big(\operatorname{Li}_2(1-cz) - \operatorname{Li}_2(1/c) + \operatorname{Li}_2(1)\Big) \\
&\qquad + \log(1-cz)\Big(\operatorname{Li}_2(1/c) - \operatorname{Li}_2(z)\Big) + \frac{1}{2}\log(c)\log^2(1-z)\Bigg].
\end{aligned}
\tag{4.3}
$$

Lewin, from whose book [15] this result was obtained, often assumes that his arguments are real; it appears that he intended the above formula to be restricted to $c \geq 1$ and $0 \leq z < 1/c$. In section 3.3, we discussed what caveats must be added to extend this to $c \in \mathbb{R}^\times$ and $z \in \mathfrak{H}$ or $-\mathfrak{H}$. But for $c \notin \mathbb{R}$, errors arise in (4.3). So we will now find an antiderivative for $\log(1-z)\log(1-cz)\frac{dz}{z}$ that works for arbitrary z and c, in part because the proofs of the theorems require it, and in part just to understand the origin of this surprisingly complicated expression. The derivation follows ideas in [15]; see also [20].

Define the function

$$\mathcal{L}(x) = \mathrm{Li}_3(x) - \log(x)\,\mathrm{Li}_2(x) - \frac{1}{2}\log(1-x)\log^2(x).$$

(Note the similarity with $P_3(x)$.) $\mathcal{L}(x)$ is holomorphic on $\mathfrak{H} \cup -\mathfrak{H} \cup (0,1)$. In what follows we will extend $\log(x)$ to $(-\infty, 0)$ by approaching from $\mathfrak{H}$, i.e., taking $-\pi < \arg(x) \leq \pi$; for $k \geq 2$, we extend $\mathrm{Li}_k(x)$ to $(1,\infty)$ by approaching from $-\mathfrak{H}$. (This convention agrees with the implementation of the principal branches found in most computer programs). As observed in (4.2),

$$\log^2(1-x)\frac{dx}{x} = d\big[-2\mathcal{L}(1-x)\big].$$

Suppose we need to integrate $\log(1-x)\log(1-cx)\frac{dx}{x}$ over a path γ; let γ° denote its interior. As we wish to use principal branches consistently, we make the following hypotheses, which prevent the functions involved from having discontinuities on γ°:

- For all $x \in \gamma^\circ$, the quantities x, cx, $\dfrac{1-x}{1-cx}$ and $\dfrac{c(1-x)}{1-cx}$ are each always in either $\mathfrak{H}$, $-\mathfrak{H}$ or $\mathbb{R}$; and

- 0, 1 and $-1/c \notin \gamma^\circ$.

E.g., both of these requirements are met if γ° does not intersect the circles/lines $\mathbb{R}$, $c^{-1}\mathbb{R}$, $\left[\begin{smallmatrix} 1 & -1 \\ c & -1 \end{smallmatrix}\right] \cdot \mathbb{R}$, and $\left[\begin{smallmatrix} 1 & -c \\ c & -c \end{smallmatrix}\right] \cdot \mathbb{R}$ (using the usual action of 2×2 matrices on $\mathbb{C}$ by linear fractional transformations). In all cases that will be of interest to us, γ can be made to meet these requirements by breaking it into a finite number of subpaths.

Pick some $x_0 \in \gamma^\circ$, and define

$$\begin{aligned} n &= \frac{1}{2\pi i}\Big[\log\Big(\frac{1-x_0}{1-cx_0}\Big) - \log(1-x_0) + \log(1-cx_0)\Big] \\ \lambda &= \log\Big(\frac{c(1-x_0)}{1-cx_0}\Big) - \log\Big(\frac{1-x_0}{1-cx_0}\Big) \qquad \Big[= \log(c) + 2\pi i m\Big]. \end{aligned}$$

n and λ are independent of the choice of $x_0 \in \gamma^\circ$. m, n are integers, and since $|\arg(z)| \le \pi$, we have that in fact $m, n \in \{-1, 0, 1\}$.

So on all of γ°,

$$\begin{aligned} \log^2\Big(\frac{1-x}{1-cx}\Big) &= \Big(\log(1-x) - \log(1-cx) + 2\pi i n\Big)^2 \\ &= \log^2(1-x) + \log^2(1-cx) - 2\log(1-x)\log(1-cx) \\ &\qquad + 4\pi i n\big(\log(1-x) - \log(1-cx)\big) + (2\pi i n)^2. \end{aligned}$$

Hence

$$\begin{aligned} \log(1-x)\log(1-cx)\frac{dx}{x} &= -\frac{1}{2}\log^2\Big(\frac{1-x}{1-cx}\Big)\frac{dx}{x} \\ &\quad -d\Big[\mathcal{L}(1-x) + \mathcal{L}(1-cx) + 2\pi i n \operatorname{Li}_2(x) \\ &\qquad - 2\pi i n \operatorname{Li}_2(cx) - \frac{1}{2}(2\pi i n)^2 \log(x)\Big]. \end{aligned}$$

It remains to integrate $\log^2\left(\frac{1-x}{1-cx}\right)$. If we let

$$u = \frac{1-x}{1-cx},$$

then $x = (1-u)/(1-cu)$. So

$$\begin{aligned}
-\frac{1}{2}\log^2\Big(\frac{1-x}{1-cx}\Big)\frac{dx}{x} &= -\frac{1}{2}\log^2(u)\, d\log\Big(\frac{1-u}{1-cu}\Big) \\
&= -\frac{1}{2}\log^2(u)\,\Big(d\log(1-u) - d\log(1-cu)\Big).
\end{aligned}$$

Let $v = cu$, so $\lambda = \log(v) - \log(u)$. Then the above becomes

$$\begin{aligned}
&-\frac{1}{2}\log^2(u)\, d\log(1-u) + \frac{1}{2}(\log(v)-\lambda)^2\, d\log(1-v) \\
&= d\big[\mathcal{L}(u)\big] + \frac{1}{2}\Big(\log^2(v) - 2\lambda\log(v) + \lambda^2\Big)\, d\log(1-v) \\
&= d\Big[\mathcal{L}(u) - \mathcal{L}(v) + \lambda\,\mathrm{Li}_2(1-v) - \frac{1}{2}\lambda^2\log(1-cx)\Big],
\end{aligned}$$

because $d\ \log(1-v) = d\ \log(1-c) - d\ \log(1-cx) = -d\ \log(1-cx)$.

In summary, for $\mathcal{L}$, γ, n and λ as described above,

$$\begin{aligned}
&\int_\gamma \log(1-x)\log(1-cx)\frac{dx}{x} \qquad (4.4)\\
&= \Bigg[-\mathcal{L}(1-x) - \mathcal{L}(1-cx) - \mathcal{L}\Big(\frac{c(1-x)}{1-cx}\Big) + \mathcal{L}\Big(\frac{1-x}{1-cx}\Big) \\
&\qquad + 2\pi i n\Big(\mathrm{Li}_2(cx) - \mathrm{Li}_2(x)\Big) + \lambda\,\mathrm{Li}_2\Big(\frac{1-c}{1-cx}\Big) \\
&\qquad\qquad + \frac{1}{2}(2\pi i n)^2\log(x) - \frac{1}{2}\lambda^2\log(1-cx)\Bigg]_{\partial\gamma}.
\end{aligned}$$

Alternatively, by doing the change of variables $(c, x) \to (c^{-1}, cx)$ in the original equation, the above is also equal to

$$\Bigg[-\mathcal{L}(1-x) - \mathcal{L}(1-cx) - \mathcal{L}\Big(\frac{1-cx}{c(1-x)}\Big) + \mathcal{L}\Big(\frac{1-cx}{1-x}\Big) + 2\pi in\Big(\mathrm{Li}_2(cx) - \mathrm{Li}_2(x)\Big) - \lambda\,\mathrm{Li}_2\Big(\frac{1-c}{-c(1-x)}\Big) + \frac{1}{2}(2\pi in)^2\log(x) - \frac{1}{2}\lambda^2\log(1-x)\Bigg]_{\partial\gamma}. \tag{4.5}$$

(The latter version is particularly convenient if you need to let $x \to 1/c$; the former is more convenient for $x \to 1$.) Similarities with (4.3) are apparent.

It is also interesting to note that the arguments of the four $\mathcal{L}$ terms in (4.4) and (4.5) are very similar to the arguments to $D(z)$ in the five-term relation (2.6); indeed, one could even insert the "missing" term $\mathcal{L}(c)$ into the antiderivatives with no ill effect, as it is a constant. The fact that one of the $\mathcal{L}$ terms appears with a different sign is puzzling, though. We will revisit this observation in chapter 5.

To ease our later discussion, we now state a few lemmas.

Lemma 4.3. *Suppose $A(z) \in \overline{\mathbb{Q}}(z)$. Then*

$$\{z \in \mathbb{C} \ : \ |z| = 1 \ \textit{ and } \ A(z) \in \mathbb{R}\}$$

is either the entire unit circle or a finite subset of $\overline{\mathbb{Q}}$.

(See proposition 4.9 for conditions under which the first possibility may occur.)

Proof. Suppose $|z| = 1$. Then $A(z)$ is real if and only if

$$A(z) = \overline{A(z)} = \overline{A}(z^{-1}),$$

which may be transformed into a polynomial equation with algebraic coefficients. The two possible outcomes in the lemma come from whether this polynomial is identically zero or not. $\square$

Lemma 4.4. *Suppose $R(z)$ and $S(z) \in \overline{\mathbb{Q}}(z)$ and $\eta = S(\tilde{\eta})$ is the image under S of some arc $\tilde{\eta}$ in the unit circle with algebraic endpoints. Then η can be decomposed into a finite number of subpaths η_j with endpoints in $\overline{\mathbb{Q}} \cup \{\infty\}$, such that for each j, $\log\big(R(z)\big)$ restricted to η_j is a fixed $\mathbb{Z}$-linear combination of functions of the form $\log(\alpha)$, $\log(z)$ and $\log(1-\alpha z)$, where $\alpha \in \overline{\mathbb{Q}}$.*

Proof. Factoring $R(z)$ as

$$c\, z^m \prod (1 - a_k z)^{m_k},$$

it is clear that we can locally decompose $\log\big(R(z)\big)$ into such linear combinations. (Note that multiples of $2\pi i$ are included, as $\pi i = \log(-1)$.) This decomposition is fixed modulo $2\pi i \mathbb{Z}$, but at those points on η where one of the logarithms is discontinuous or the path goes off to ∞, the multiple of $2\pi i$ may change; the path η needs to be broken apart at those points.

The only question is whether the finite "break points" are algebraic. The finite break points come in two varieties: poles of $R(z)$ (which are obviously algebraic), and $z \in \eta$ such that $R(z)$, z or $1 - \alpha_k z$ cross through $\mathbb{R}$. We apply the previous lemma with $A(w) = R \circ S(w)$, $S(w)$ or $1 - \alpha_k S(w)$. If

$\{w \,:\, |w| = 1 \text{ and } A(w) \in \mathbb{R}\}$ is finite, the claim follows. If it is the whole unit circle, then it is only necessary to make breaks where $A(w) = 0$, and these points are algebraic. $\square$

Lemma 4.5. *Suppose $f(z)$, $g(z)$, $h(z)$ and $S(z) \in \overline{\mathbb{Q}}(z)$ and $\eta = S(\tilde{\eta})$ is the image under S of some arc $\tilde{\eta}$ in the unit circle with algebraic endpoints. Then*

$$\int_\eta \log(f) \log(g)\, d\log(h) \;\in\; \mathcal{S},$$

$$\int_\eta \mathrm{Li}_2(f)\, d\log(g) \;\in\; \mathcal{S},$$

and

$$\int_\eta \log(f)\, d\log(g) \;\in\; \mathcal{T}.$$

Proof. For the first assertion, factor $h(z)$ as

$$c \prod_{j=1}^{k} (z - \alpha_j)^{m_j},$$

for c, $\alpha_j \in \overline{\mathbb{Q}}$, $m_j \in \mathbb{Z}$. Then

$$\int_\eta \log(f) \log(g)\, d\log(h) = \sum_{j=1}^{k} m_j \int_\eta \log(f) \log(g)\, \frac{dz}{z - \alpha_j}.$$

So performing a linear change of variables, we reduce to the case $h(z) = z$. Invoking lemma 4.4 , we further reduce to the case where $f(z)$ and $g(z)$ belong to the set

$$\overline{\mathbb{Q}} \cup \{z\} \cup \{1 - \alpha z \,:\, \alpha \in \overline{\mathbb{Q}}\}.$$

Now, by an examination of equations (4.2) and (4.4), the first assertion of the lemma follows. (Any additional break points needed to use equation (4.4) are also algebraic.) The proof of the third assertion is similar.

As for the second assertion, we break up the path η at those z where $\mathrm{Li}_2\big(f(z)\big)$ or $\log\big(g(z)\big)$ is discontinuous; as in the proof of lemma 4.4, these are all algebraic. Then integrating by parts,

$$\int_\eta \mathrm{Li}_2(f)\, d\log(g) = \Big[\mathrm{Li}_2(f)\log(g)\Big]_{\partial\eta} + \int_\eta \log(g)\log(1-f)\, d\log(f),$$

and both terms on the right side belong to $\mathcal{S}$. □

4.3 The first family

Here we will prove theorem 4.1. Let us recall the statement:

Theorem. *For $R(z) \in \overline{\mathbb{Q}}(z)$ equal to the product of a reciprocal polynomial of even degree with the square of a rational function,*

$$m\big(1 + x + R(z)(1-x)y\big) \;\in\; \frac{1}{\pi^2}\,\mathcal{S}.$$

To show this, let us reexamine the proof of theorem 3.3. There it was shown that, for the specific rational function $R(z) = z + z^{-1}$,

$$\pi^2\, m\big(1 + x + R(z)(1-x)y\big) \;=\; -\,\mathrm{Re}\left[\int_{|z|=1} \mathrm{Li}_2\big(i|R(z)|\big)\,\frac{dz}{z}\right]. \tag{4.6}$$

The next step in the argument was to observe that, on $C_\pm$,

$$|R(z)| = \mp i\, R(z),$$

i.e. $|R(z)|$ is itself (locally) equal to a rational function when restricted to $|z| = 1$. So by breaking up the path, the absolute values may be eliminated and the integrand in (4.6) becomes holomorphic. The idea for this first family of examples is that this strategy works for other choices of $R(z)$. To isolate exactly which choices do work, we need

Lemma 4.6. *For $R(z) \in \mathbb{C}(z)$, $|R(z)|$ is locally equal to a rational function everywhere on the unit circle if and only if $R(z)$ is the product of a reciprocal polynomial of even degree, a power of z, and the square of a rational function.*

Further, if these conditions are met, then $|R(z)|$ is equal to the same rational function, up to sign, on the whole unit circle.

(By "locally equal to a rational function," we mean that there exists an arc γ in the unit circle and $A_\gamma(z) \in \mathbb{C}(z)$ such that, restricted to γ, $|R(z)| \equiv A_\gamma(z)$. By "everywhere", we mean that we can cover the unit circle with such arcs.)

Before proving this lemma, let us show how the theorem follows from it. Suppose $R(z)$ is a rational function meeting the hypotheses of the theorem.[1] By the above lemma, there exists a single rational function $A(z)$ such that $i\,|R(z)| = \pm A(z)$ everywhere on the unit circle. Since

$$A^2(z) = \big(i\,|R(z)|\big)^2 = -R(z)\,\overline{R}(z^{-1})$$

on the unit circle (thus in all on $\mathbb{C}$), $A^2(z)$ is in $\overline{\mathbb{Q}}(z)$; it follows that $A(z) \in \overline{\mathbb{Q}}(z)$ also.

So (4.6) becomes

$$\pi^2\, m\big(1 + x + R(z)(1-x)y\big) \;=\; -\operatorname{Re}\left[\sum_{\pm}\int_{\gamma_\pm} \operatorname{Li}_2\big(\pm A(z)\big)\,\frac{dz}{z}\right], \qquad (4.7)$$

where we have divided the unit circle into arcs $\gamma_\pm$ according to whether $i\,|R(z)| = \pm A(z)$. The endpoints of these arcs are zeros or poles of $R(z)$,

[1] In the hypotheses of the theorem, we have omitted the possibility of a power of z in $R(z)$. This is because such a power could be absorbed into y by a change of variables.

therefore algebraic. So by lemma 4.5, the contents of the brackets on the right side above belong to $\mathcal{S}$. Because $\overline{\mathrm{Li}_k(z)} = \mathrm{Li}_k(\bar{z})$ and $\overline{\mathrm{Li}_k^+(r)} = \mathrm{Li}_k^-(r)$ for $r \in \mathbb{R}$, it follows that for every $s \in \mathcal{S}$, $\bar{s} \in \mathcal{S}$ also, so

$$\mathrm{Re}[s] = \frac{1}{2}(s + \bar{s}) \in \mathcal{S}.$$

This completes the proof of theorem 4.1.

We now prove the lemma.

Proof. One direction is easy: If $P(z) \in \mathbb{C}[z]$ is reciprocal of degree $2n$, then on $|z| = 1$,

$$|P(z)|^2 = P(z)\overline{P}(z^{-1}) = z^{-2n}P(z)\,P^\star(z) = (z^{-n}P(z))^2,$$

hence $|P(z)| = \pm z^{-n}P(z) \in \mathbb{C}(z)$. Of course, $|z^k| = 1$, and for $S(z) \in \mathbb{C}(z)$,

$$|S(z)^2| = |S(z)|^2 = S(z)\overline{S}(z^{-1}) \in \mathbb{C}(z).$$

For the converse, observe that $R(z)$ may be factored as

$$R(z) = z^k \cdot S^2(z) \cdot P(z),$$

where $k \in \mathbb{Z}$, $S(z) \in \mathbb{C}(z)$, and $P(z)$ is a squarefree polynomial with $P(0) \neq 0$. If $|R(z)|$ is locally equal to a rational function on the unit circle, then so is $|P(z)|$; suppose that $|P(z)| = A(z) \in \mathbb{C}(z)$ on some subarc of $|z| = 1$. Then if $d = \deg(P)$,

$$A^2(z) = |P(z)|^2 = P(z)\overline{P}(z^{-1}) = z^{-d}P(z)\,P^\star(z).$$

As $A^2(z)$ and $z^{-d}P(z)\,P^\star(z)$ are both meromorphic functions that match on an arc, they must be identically equal in all of $\mathbb{C}$. (In particular, the only other rational function that $|P(z)|$ could be equal to on the unit circle is $-A(z)$, which proves the second part of the lemma.) Further, since z divides neither P nor $P^\star$, the order of the pole of $A^2(z)$ at zero is d, so d must be even.

Since P (and hence also $P^\star$) is squarefree and $P(z)\,P^\star(z) = z^d\,A^2(z)$ is a square, every root of P must be a root of $P^\star$, and vice versa. Hence

$$P^\star(z) = \alpha\,P(z)$$

for some constant α. Since $P = P^{\star\star} = \bar{\alpha}\,P^\star = |\alpha|^2 P$, we must have $|\alpha| = 1$, so we can write $\alpha = \beta^2$, where $|\beta| = 1$ also. Let $P_1 = \beta\,P$. Then

$$P_1^\star = \bar{\beta}\;P^\star = \bar{\beta}\alpha P = (\bar{\beta}\beta)\cdot\beta P = P_1.$$

Hence $P_1(z)$ is reciprocal of even degree, so

$$R(z) = z^k \cdot \left(\beta^{-1/2}\,S(z)\right)^2 \cdot P_1(z)$$

is of the expected form. □

4.4 The second family

Here we will prove theorem 4.2. Notice that to obtain theorem 4.1, we started with a preexisting Mahler measure evaluation in terms of polylogarithms (theorem 3.2) and replaced a parameter in that formula with a rational function. Theorem 4.2 will come about in a similar manner, originating from the following result of Maillot and Cassaigne [17].

Theorem 4.7. *For any $a_1, a_2, a_3 \in \mathbb{C}^\times$,*

$$\pi\, m(a_1x_1 + a_2x_2 + a_3x_3) = \begin{cases} D\left(\left|\dfrac{a_1}{a_3}\right| e^{i\alpha_2}\right) + \displaystyle\sum_{k=1}^{3} \alpha_k \log|a_k| & \text{if "}\triangle\text{"}, \\ \pi \log \max\{|a_1|, |a_2|, |a_3|\} & \text{otherwise.} \end{cases}$$

Here, the condition "$\triangle$" means that $|a_1|$, $|a_2|$ and $|a_3|$ are the lengths of the sides of some triangle, and in that case, α_k is the radian measure of the angle opposite the side of length $|a_k|$. $D(z)$, as before, is the Bloch-Wigner dilogarithm

$$D(z) = P_2(z) = \mathrm{Im}\big[\mathrm{Li}_2(z)\big] + \log|z| \arg(1-z).$$

(Although $m(a_1x_1 + a_2x_2 + a_3x_3)$ is symmetric in a_1, a_2 and a_3, the expression on the right side in the theorem appears not to be; this apparent problem is explained away by the many functional equations satisfied by $D(z)$.)

We will not use the full power of this theorem, but only a special case. If $a_1 = c$, $a_2 = 1-c$ and $a_3 = 1$, then we are automatically in the "$\triangle$" case, although the triangle is degenerate if $c \in \mathbb{R}$. Observing then that $\alpha_1 = -\operatorname{sign}\big(\mathrm{Im}[c]\big) \cdot \arg(1-c)$ and $\alpha_2 = \operatorname{sign}\big(\mathrm{Im}[c]\big) \cdot \arg(c)$ and using corollary 2.3, we have

Corollary 4.8. *For $c \in \mathbb{C} \setminus \mathbb{R}$,*

$$m\big(cx + (1-c)y + 1\big) = \frac{\operatorname{sign}\big(\mathrm{Im}[c]\big)}{\pi} \Big(\mathrm{Im}\big[\mathrm{Li}_2(c)\big] + \log|1-c| \arg(c)\Big).$$

And for $c \in \mathbb{R}$,

$$m\big(cx + (1-c)y + 1\big) = \begin{cases} \log(1-c) & \text{if } c < 0, \\ 0 & \text{if } 0 \le c \le 1, \\ \log(c) & \text{if } c > 1. \end{cases}$$

(The $c \in \mathbb{R}$ cases may be proved directly by use of Jensen's formula in one of the variables, or by taking limits in the $c \in \mathbb{C} \setminus \mathbb{R}$ formula, using theorem 2.4.)

Observe that the polynomial in the corollary may be replaced with a slightly more attractive one:

$$m\bigl(cx+(1-c)y+1\bigr) = m(y)+m\bigl(y^{-1}+1-c(-y^{-1}x+1)\bigr) = m\bigl(x+1-c(y+1)\bigr),$$

where the second equality follows from propositions 2.1 and 2.2.

We are now ready to prove theorem 4.2, which we recall here:

Theorem. *For any rational function $R(z) \in \overline{\mathbb{Q}}(z)$,*

$$m\bigl(x+1-R(z)(y+1)\bigr) \in \begin{cases} \dfrac{1}{\pi i}\,\mathcal{T} & \textit{if } R(z) \textit{ maps the unit circle into } \mathbb{R}, \\ \dfrac{1}{\pi^2}\,\mathcal{S} & \textit{otherwise.} \end{cases}$$

Proof. We first deal with the second case, where $R(z)$ does not map the unit circle entirely into $\mathbb{R}$. Break the unit circle into arcs $\gamma_\pm$ depending on the sign of $\mathrm{Im}\bigl[R(z)\bigr]$; by lemma 4.3, the endpoints are algebraic. Then by the above corollary,

$$\begin{aligned} &m\bigl(x+1-R(z)(y+1)\bigr) \\ &\qquad = \sum_{\pm} \frac{\pm 1}{2\pi i}\frac{1}{\pi}\int_{\gamma_\pm}\Bigl(\mathrm{Im}\bigl[\mathrm{Li}_2(R)\bigr] + \log|1-R|\cdot\mathrm{Im}\bigl[\log(R)\bigr]\Bigr)\frac{dz}{z}. \end{aligned}$$

Since $dz/(iz) = d\arg(z)$ is real, this equals

$$\begin{aligned} &\sum_{\pm}\frac{\pm 1}{2\pi^2}\,\mathrm{Im}\left[\int_{\gamma_\pm}\Bigl(\mathrm{Li}_2(R) + \log|1-R|\,\log(R)\Bigr)\frac{1}{i}\frac{dz}{z}\right] \\ &= \sum_{\pm} \mp\,\mathrm{Re}\left[\frac{1}{2\pi^2}\int_{\gamma_\pm}\Bigl(\mathrm{Li}_2(R) + \frac{1}{2}\bigl(\log(1-R)+\log(1-\overline{R}(z^{-1}))\bigr)\log(R)\Bigr)\frac{dz}{z}\right]. \end{aligned}$$

And by lemma 4.5, this belongs to $\dfrac{1}{\pi^2}\mathcal{S}$.

Now suppose that $R(z)$ does map the unit circle entirely into $\mathbb{R}$. Then by corollary 4.8,

$$m\big(x+1-R(z)(y+1)\big) = \frac{1}{2\pi i}\left[\int_{\gamma_1} \log\big(R(z)\big)\frac{dz}{z} + \int_{\gamma_2} \log\big(1-R(z)\big)\frac{dz}{z}\right],$$

where $\gamma_1 = \{z : |z| = 1 \text{ and } R(z) > 1\}$ and $\gamma_2 = \{z : |z| = 1 \text{ and } R(z) < 0\}$. At the endpoints of γ_1 and γ_2, $R(z)$ is either equal to 0 or 1 or has a pole, so they are algebraic. Thus by lemma 4.5, the above belongs to $\dfrac{1}{\pi i}\mathcal{T}$. $\square$

Recall that in the hypotheses of the theorem, we make a distinction between whether image of the unit circle under our rational function lies inside of $\mathbb{R}$ or not. To clarify just what this means, we offer the following

Proposition 4.9. *A rational function in $\mathbb{C}(z)$ maps the unit circle into $\mathbb{R}$ if and only if it is of the form*

$$z^k\,\frac{P(z)}{Q(z)},$$

where P and Q are reciprocal polynomials, $\deg(P)$ *and* $\deg(Q)$ *have the same parity, and $k = \frac{1}{2}\big(\deg(Q) - \deg(P)\big)$.*

Proof. Suppose $R(z) \in \mathbb{C}(z)$ maps the unit circle to $\mathbb{R}$. Write

$$R(z) = z^k\,\frac{P(z)}{Q(z)},$$

where P, Q are relatively prime polynomials not vanishing at $z = 0$, and $k \in \mathbb{Z}$. Let $d = \deg(P)$, $e = \deg(Q)$. So on the unit circle, $R(z) = \overline{R(z)}$, i.e.

$$z^k\,\frac{P(z)}{Q(z)} = z^{-k}\,\frac{\overline{P}(z^{-1})}{\overline{Q}(z^{-1})},$$

hence

$$z^{k+d}\,P(z)\,Q^\star(z) = z^{e-k}\,P^\star(z)Q(z).$$

None of P, $P^\star$, Q or $Q^\star$ are divisible by z, so $2k = e - d$ (which implies that $d \equiv e \mod 2$) and

$$\frac{P(z)}{Q(z)} = \frac{P^\star(z)}{Q^\star(z)}. \tag{4.8}$$

Since P and Q are relatively prime, it follows that $P \mid P^\star$; since they have the same degree, $P^\star = \alpha P$ for some constant α. Likewise, $Q^\star = \alpha Q$ (the constant must be the same for both). As in the proof of lemma 4.6, if we write $\alpha = \beta^2$ and define

$$P_1 = \beta\,P \qquad \text{and} \qquad Q_1 = \beta\,Q,$$

then P_1 and Q_1 are both reciprocal, and the claim follows.

We omit the proof of the converse, as it is quite direct. □

As observed earlier, theorems 4.1 and 4.2 were produced from preexisting Mahler measure evaluations by replacing a parameter with a rational function. It is clear that other Mahler measure evaluations in terms of logarithms and dilogarithms could produce other examples. For instance, Vandervelde's formula in [24] is promising, although as with the Maillot-Cassaigne formula (which it generalizes), some cleverness would be required to handle the angles in the formula.

4.5 An example

In this section we will work out one example in more detail. Define

$$P_{a,b,c,d}(x,y,z) = (az+b)(x+1) - (cz+d)(y+1).$$

We will denote the Mahler measure of $P_{a,b,c,d}$ by $m(a,b,c,d)$. Because

$$\begin{aligned} m(a,b,c,d) &= m(cz+d) \ + \ m\Big(\Big(\frac{az+b}{cz+d}\Big)(x+1) - (y+1)\Big) \\ &= \log\max\{|c|,|d|\} \ + \ m\Big(y+1-\Big(\frac{az+b}{cz+d}\Big)(x+1)\Big), \end{aligned}$$

this can be handled by Theorem 4.2. In this case, the rational function R in the theorem is the linear fractional transformation

$$\frac{az+b}{cz+d} = M\cdot z, \quad \text{where} \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In principle, very few hypotheses need to be made on the four parameters a, b, c and d, but the calculation is greatly simplified if we impose some restrictions. Henceforth we will assume that a, b, c $d \in \mathbb{R}^\times$, that $\{a,d\} \cap \{b,c\} = \varnothing$, and that $ad - bc \neq 0$ (so M gives a Möbius transformation).[2] Proposition 4.9 then guarantees that $m\big(y+1-(M\cdot z)(x+1)\big)$ will belong to $\mathcal{S}/\pi^2$.

Throughout, we will use w to denote $M \cdot z$. Let us also use the shorthand $D = ad - bc$ and $\mathrm{s}_r := \operatorname{sign}(r)$ for $r \in \mathbb{R}^\times$. Recall that with Möbius transformations, ignoring poles,

$$z \in \pm\mathfrak{H} \quad \Longleftrightarrow \quad w \in \pm\mathrm{s}_D\,\mathfrak{H}.$$

[2] We will regard four-tuples (a,b,c,d) which fail either of the last two hypotheses as degenerate cases. If it is desired to evaluate one of these cases, one approach would simply be to take a limit in the formula for the nondegenerate cases, using theorem 2.4.

Further, the roots and poles of $\frac{az+b}{cz+d}$ are real, so it will only be necessary to break up the unit circle at ± 1. We split the unit circle into the two arcs

$$C_{\pm} = \{z \,:\, |z| = 1 \text{ and } z \in \pm\mathfrak{H}\}.$$

We may write

$$\begin{aligned}
\mu &:= m(a,b,c,d) - \log\max\{|c|,|d|\} \\
&= m\big(w(x+1)-(y+1)\big) \\
&= \frac{1}{2\pi i}\int_{|z|=1} m_{x,y}\big(w(x+1)-(y+1)\big)\frac{dz}{z},
\end{aligned} \tag{4.9}$$

where, as before, $m_{x,y}$ means that x and y are the variables in that Mahler measure (w is treated as a constant). But by proposition 2.2,

$$m_{x,y}\big(w(x+1)-(y+1)\big) = m_{x,y}\big(w(x^{-1}+1)-(y^{-1}+1)\big).$$

So since

$$\big|w(x^{-1}+1)-(y^{-1}+1)\big| = \big|\overline{\bar{w}(x+1)-(y+1)}\big| = \big|(M\cdot z^{-1})(x+1)-(y+1)\big|,$$

it follows that the integrals over C_+ and C_- are redundant, and (4.9) becomes

$$\mu = \frac{1}{\pi i}\int_{C_+} m_{x,y}\big(w(x+1)-(y+1)\big)\frac{dz}{z}. \tag{4.10}$$

Define k_1, k_2 and l to be the unique integers such that for all $z \in \mathfrak{H}$ (or equivalently, for all $w \in \mathrm{s}_D\,\mathfrak{H}$),

$$\begin{aligned}
\log(M\cdot z) &= \log\Big(1+\frac{a}{b}z\Big) - \log\Big(1+\frac{c}{d}z\Big) + \log|b/d| + k_1\pi i \\
\log(M\cdot z^{-1}) &= \log\Big(1+\frac{b}{a}z\Big) - \log\Big(1+\frac{d}{c}z\Big) + \log|a/c| + k_2\pi i \\
\log(M^{-1}\cdot w) &= \log\Big(1-\frac{d}{b}w\Big) - \log\Big(1-\frac{c}{a}w\Big) + \log|b/a| + l\pi i
\end{aligned} \tag{4.11}$$

Also define $k = k_1 - k_2$. Letting $z \to i\infty$ for k_1 and k_2 and letting $w \to \mathrm{s}_D \cdot i\,\infty$ for l, it may be seen that in fact

$$k_1 = \begin{cases} \frac{1}{2}\,\mathrm{s}_a\,(\mathrm{s}_d - \mathrm{s}_b) & \text{if } \mathrm{s}_a = \mathrm{s}_c, \\ \mathrm{s}_D - \frac{1}{2}\,\mathrm{s}_a\,(\mathrm{s}_b + \mathrm{s}_d) & \text{if } \mathrm{s}_a \neq \mathrm{s}_c, \end{cases}$$

$$k_2 = \begin{cases} \frac{1}{2}\,\mathrm{s}_b\,(\mathrm{s}_c - \mathrm{s}_a) & \text{if } \mathrm{s}_b = \mathrm{s}_d, \\ -\mathrm{s}_D - \frac{1}{2}\,\mathrm{s}_b\,(\mathrm{s}_a + \mathrm{s}_c) & \text{if } \mathrm{s}_b \neq \mathrm{s}_d, \end{cases}$$

$$l = \begin{cases} 1 + \frac{1}{2}\,\mathrm{s}_D\,\mathrm{s}_c\,(\mathrm{s}_b - \mathrm{s}_a) & \text{if } \mathrm{s}_c = \mathrm{s}_d, \\ -\frac{1}{2}\,\mathrm{s}_D\,\mathrm{s}_c\,(\mathrm{s}_a + \mathrm{s}_b) & \text{if } \mathrm{s}_c \neq \mathrm{s}_d. \end{cases}$$

Roughly following the proof of theorem 4.2, (4.10) gives us

$$-\mathrm{s}_D\,\pi^2\,\mu = \mathrm{Re}\left[\int_{C_+} \mathrm{Li}_2(w)\frac{dz}{z} \;+\; \int_{C_+} \log|1-w|\log(w)\,\frac{dz}{z}\right]. \tag{4.12}$$

Integrating by parts, the first integral is equal to

$$\mathrm{Re}\left[\Big[\mathrm{Li}_2(w)\log(z)\Big]_{\partial C_+} + \int_{C_+} \log(z)\log(1-w)\,d\log(w)\right].$$

In this integral, w is a function of z, and we are integrating with respect to z. If we change variables and integrate with respect to w, this becomes

$$\mathrm{Re}\left[\pi i\;\mathrm{Li}_2^{\mathrm{s}_D}\Big(\frac{a-b}{c-d}\Big) + \int_{M\cdot C_+} \log\big(M^{-1}\cdot w\big)\log(1-w)\frac{dw}{w}\right], \tag{4.13}$$

where $\mathrm{Li}_2^{\mathrm{s}_D}$ refers to Li_2^+ or Li_2^-, depending on s_D.

For the second integral in (4.12), first note that we may expand

$$\log|1-w| = \log\left|1 + \frac{a-c}{b-d}z\right| - \log\left|1 + \frac{c}{d}z\right| + \log\left|\frac{b-d}{d}\right|. \tag{4.14}$$

Now, for any expression u (a function of z),

$$\begin{aligned}
\mathrm{Re}\biggl[\int_{C_+} \log|u| \log(w)\,\frac{dz}{z}\biggr] &= \int_{C_+} \log|u|\;\mathrm{Im}\bigl[\log(w)\bigr]\, i\frac{dz}{z} \\
&= \mathrm{Re}\biggl[\int_{C_+} \log(u)\;\mathrm{Im}\bigl[\log(w)\bigr]\, i\frac{dz}{z}\biggr] \\
&= \mathrm{Re}\biggl[\int_{C_+} \log(u)\,\frac{\log(M\cdot z) - \log(M\cdot z^{-1})}{2}\,\frac{dz}{z}\biggr].
\end{aligned}$$

Combining the above with (4.11), (4.13) and (4.14), we obtain

$$\begin{aligned}
-\mathrm{s}_D\,\pi^2\mu = \mathrm{Re}\biggl[&\pi i\;\mathrm{Li}_2^{\mathrm{s}_D}\Bigl(\frac{a-b}{c-d}\Bigr) + \bigl(\log|b/a| + l\pi i\bigr)\cdot K_{M\cdot C_+}(1) \\
&+ J_{M\cdot C_+}(1, d/b) - J_{M\cdot C_+}(1, c/a) \\
&+ \frac{1}{2}\,J_{C_+}(A\cdot B) + \frac{\pi i}{2}\log\Bigl|\frac{b-d}{d}\Bigr|\,\Bigl(\log\Bigl|\frac{bc}{ad}\Bigr| + k\pi i\Bigr) \\
&+ \frac{1}{2}\log\Bigl|\frac{b-d}{d}\Bigr|\,K_{C_+}(B) + \frac{1}{2}\Bigl(\log\Bigl|\frac{bc}{ad}\Bigr| + k\pi i\Bigr)\,K_{C_+}(A)\biggr],
\end{aligned} \tag{4.15}$$

where we have adopted the following notation:

$$\begin{aligned}
J_\gamma(\alpha,\beta) &= \int_\gamma \log(1-\alpha x)\log(1-\beta x)\frac{dx}{x}, \\
K_\gamma(\alpha) &= \int_\gamma \log(1-\alpha x)\frac{dx}{x}, \\
A &= \Bigl[-\Bigl(\frac{a-c}{b-d}\Bigr)\Bigr] - \Bigl[-\frac{c}{d}\Bigr], \\
B &= [-a/b] - [-b/a] - [-c/d] + [-d/c].
\end{aligned}$$

(A and B are elements of the group ring on the free group generated by elements of $\mathbb{R}$. J_γ and K_γ are extended by linearity over such sums that are homogeneous of degree two and one, respectively.)

Note that the $\frac{\pi i}{2}\log\left|\frac{b-d}{d}\right|\,\log\left|\frac{bc}{ad}\right|$ term in (4.15) may be dropped, as its real part is zero.

For a matrix $A \in M_2(\mathbb{R})$ and $\alpha \in \mathbb{R}$, let $s = \operatorname{sign}\big(\alpha \cdot \det(A)\big)$. Then the interior of the path $\alpha\,(A \cdot C_+)$ lies in $s\,\mathfrak{H}$, with endpoints in $\mathbb{R}$. So

$$K_{A\cdot C_+}(\alpha) = K_{\alpha\,(A\cdot C_+)}(1) = -\operatorname{Li}_2^s\big(\alpha\,(A\cdot\ -1)\big) + \operatorname{Li}_2^s\big(\alpha\,(A\cdot 1)\big).$$

For $\alpha, \beta \neq 0$, $J_\gamma(\alpha,\beta) = J_{\alpha\cdot\gamma}(1,\beta/\alpha) = J_{\beta\cdot\gamma}(1,\alpha/\beta)$. Consider the two integrals of form J_γ in (4.15); The constants involved in them are real, and the interiors of the paths of integration lie entirely in one half-plane and have endpoints in $\mathbb{R}$. Therefore, they may be evaluated by formula (3.10) (as opposed to the more general but unwieldy (4.4) or (4.5)). In fact, since we only need the real part, this may be cleaned up further. For $\nu = \pm 1$, let us denote

$$\{w\}_\nu \;=\; \begin{cases} w & \text{if } \nu = +1, \\ \bar{w} & \text{if } \nu = +1. \end{cases}$$

We have the following restatement of (3.10):

Proposition 4.10. *Suppose $c \in \mathbb{R}^\times$ and γ is a path from α to β (both real numbers) whose interior lies entirely in $\mathfrak{H}$ or $-\mathfrak{H}$. Let $\delta = \operatorname{sign}(c)$. Then*

$$\operatorname{Re}\left[\int_\gamma \log(1-x)\log(1-cx)\frac{dx}{x}\right] \;=\; F_c(b) - F_c(a),$$

where $F_c(x)$ is defined to be

$$\begin{aligned}\operatorname{Re}\Bigg[&\operatorname{Li}_3\Big(\frac{1-cx}{1-x}\Big) \;-\; \operatorname{Li}_3(1-cx) \;-\; \operatorname{Li}_3(1-x) \;-\; \operatorname{Li}_3\Big(\frac{1-cx}{c(1-x)}\Big) \\ &+\; \log(1-x)\Big[\big\{\operatorname{Li}_2(1-cx)\big\}_{-\delta} \;-\; \operatorname{Li}_2(1/c) + \operatorname{Li}_2(1)\Big] \\ &+\; \big\{\log(1-cx)\big\}_\delta\Big[\operatorname{Li}_2(1/c) - \operatorname{Li}_2(x)\Big] \;+\; \frac{1}{2}\log(c)\log^2(1-x)\Bigg],\end{aligned}$$

and we use the standard conventions[3]*:*

$$\begin{aligned}\mathrm{Li}_2(1/c) &= \mathrm{Li}_2^-(1/c)\\ \log(c) &= \lim_{z\to c}{}^+ \log(z).\end{aligned}$$

We omit the proof, as it is relatively direct.

The above results allow us to express $m(a,b,c,d)$ as an explicit expression in terms of polylogarithms of elements of $\mathbb{Q}(a,b,c,d)$. We have written a program in Mathematica that performs this computation. For comparison, the program also evaluates $m(a,b,c,d)$ by numerically integrating the expression in (4.12). For each of the hundreds of nondegenerate tuples (a,b,c,d) we have tested, the two calculations have agreed to at least 7 decimal places. The code may be found at:

`http://www.ma.utexas.edu/users/jcondon/writings.html`

Unfortunately, the polylogarithm evaluations for $m(a,b,c,d)$ are enormously complex, having an average of about 150 terms. A certain amount of simplification might be obtained by use of identities, especially in degenerate cases. But in general, large expressions are probably unavoidable.

[3]These are the conventions used by most computer packages for calculating polylogarithms of real numbers.

Chapter 5

An algebraic approach

At the end of chapter 3, we observed some hints that the proof of proposition 3.1 might also be accomplished with the use of the modified trilogarithms $P_3(x)$ instead of the usual versions. Such a proof would be desirable for a number of reasons. First of all, because $P_3(x)$ is continuous on all of $\mathbb{P}^1(\mathbb{C})$, we no longer need to be so cautious about branch cuts. Second, the polylogarithm identities satisfied by $P_3(x)$ do not involve any lower weight terms, so a proof along these lines would resolve the mystery as to why the logarithm and dilogarithm terms in equation (3.14) completely disappear in the end.

But there is a deeper reason why we would like such an alternative proof. The function $P_3(x)$ is tied up in a construction with connections to algebraic K-theory. This setup allows for the possibility of calculating our Mahler measure value by algebraic means, providing a more intrinsic explanation for the existence of our identity, and making it look less like a coincidence of calculus. Indeed, a number of calculations along these lines have been done for other polynomials by Boyd and Rodríguez Villegas [5], [6] and by Lalín [13]. The general idea is due to Deninger [7], with the details having been worked by Rodríguez Villegas [25] and Boyd.

5.1 The two-variable case

In order to understand the construction for the case of three-variable polynomials better, we will first describe the situation for polynomials in two variables.

Let $C : P(x, y) = 0$ be a smooth projective curve, for $P(x, y) \in \mathbb{C}[x, y]$. For rational functions $f,\, g \in \mathbb{C}(C)^{\times}$, we define

$$\eta(f, g) = \log|f|\, d\arg(g) - \log|g|\, d\arg(f).$$

This is a real, C^{∞} 1-form on $C \setminus S$, where S is the set of zeros and poles of f and g. (Although $\arg(f)$ is not globally defined, $d\arg(f) = \operatorname{Im}\big[df/f\big]$ is.) It is easily checked that η is skew-symmetric and bi-additive, i.e.

$$\begin{aligned} \eta(f, g) &= \; -\eta(g, f), \\ \eta(f_1 f_2, g) &= \eta(f_1, g) + \eta(f_2, g). \end{aligned} \tag{5.1}$$

Further, because C has dimension 1,

$$d\eta = \operatorname{Im}\left[\frac{df}{f} \wedge \frac{dg}{g}\right] = 0,$$

hence η is closed. Therefore, we may associate to $\eta(f, g)$ an element $r(f, g)$ of $H^1(C \setminus S,\, \mathbb{R})$ by the mapping

$$[\gamma] \;\mapsto \int_{\gamma} \eta(f, g)$$

for a class $[\gamma] \in H_1(C \setminus S, \mathbb{Z})$. (We identify $H^1(C \setminus S,\, \mathbb{R})$ with the dual space $H_1(C \setminus S,\, \mathbb{Z})^{*}$.)

If $f \not\equiv 1$, then $\eta(f, 1 - f)$ turns out to be exact:

$$\eta(f, 1 - f) = d\,D(f), \tag{5.2}$$

where D is again the Bloch-Wigner dilogarithm. Hence,

$$r(f, 1-f) = 0. \tag{5.3}$$

Let $\Omega^n(C)$ denote the space of real 1-forms on C that are C^∞ except for isolated singularities. The properties in (5.1) imply that η, as a map from $\mathbb{C}(C)^\times \times \mathbb{C}(C)^\times$ to $\Omega^1(C)$, factors through $\Lambda^2\big(\mathbb{C}(C)^\times\big)$. Further, as η maps to a real vector space, it kills torsion in $\Lambda^2\big(\mathbb{C}(C)^\times\big)$, and we have a well-defined map[1]

$$\eta:\ \Lambda^2\big(\mathbb{C}(C)^\times\big) \otimes \mathbb{Q} \ \to\ \Omega^1(C).$$

Viewing P as a polynomial in x with coefficients in $\mathbb{C}(y)$, we may factor it as

$$P(x,y) = a_0(y) \prod_{j=1}^{n} \big(x - x_j(y)\big),$$

where $a_0(y)$ is a polynomial, and for each j, $x_j(y)$ is an algebraic function of y. By Jensen's formula,

$$m(P) = m\big(a_0(y)\big) + \sum_{j=1}^{n} \frac{1}{2\pi i} \int_{|y|=1} \log^+\!\big|x_j(y)\big| \, \frac{dy}{y}. \tag{5.4}$$

For each j, the set

$$\gamma_j = \Big\{ \big(x_j(y), y\big) \ :\ |y| = 1 \text{ and } |x_j(y)| \geq 1 \Big\}$$

[1]Similarly, r maps to a direct limit over finite subsets $S \subset C$:

$$r:\ \Lambda^2\big(\mathbb{C}(C)^\times\big) \ \to\ \varinjlim_{S} H^1(C \setminus S, \mathbb{R}).$$

Because of (5.3), this map factors through the group $K_2\big(\mathbb{C}(C)\big) = \Lambda^2\big(\mathbb{C}(C)^\times\big)/\langle f \wedge (1-f)\rangle$; the map thus obtained is known as a regulator. We will discuss K_2 more in section 5.3.

is a directed path (or a union of such) inside of C. The set $\bigcup \gamma_j$ precisely coincides with

$$\gamma = \big\{(x,y) \in C \,:\, |y| = 1,\ |x| \geq 1\big\}.$$

Therefore (5.4) may be rewritten as

$$\begin{aligned} m(P) &= m\big(a_0(y)\big) + \frac{1}{2\pi i} \int_\gamma \log|x| \, \frac{dy}{y} \\ &= m\big(a_0(y)\big) + \frac{1}{2\pi} \int_\gamma \eta(x,y). \end{aligned} \tag{5.5}$$

Thinking of x, y as elements of $\mathbb{C}(C)$, suppose we find a decomposition

$$x \wedge y \ = \ \sum_j a_j \cdot z_j \wedge (1 - z_j) \tag{5.6}$$

in $\Lambda^2\big(\mathbb{C}(C)^\times\big) \otimes \mathbb{Q}$. Then

$$\eta(x,y) \ = \ \sum_j a_j \, \eta(z_j, 1-z_j) \ = \ d\,D\left(\sum_j a_j\,[z_j]\right).$$

(We extend D over such sums by linearity.) Thus $\eta(x,y)$ is exact, and we may evaluate (5.5) using Stokes' Theorem.

In summary, evaluating $m(P)$ by this technique amounts to finding a decomposition of the form (5.6), or equivalently, to showing that $\{x,y\} = 0$ in $K_2\big(\mathbb{C}(C)\big) \otimes \mathbb{Q}$.

5.2 The three-variable case

Now suppose that we have a smooth projective surface $S : P(x,y,z) = 0$, for $P(x,y,z) \in \mathbb{C}[x,y,z]$ a polynomial relatively prime to its reciprocal polynomial

$P^\star(x,y,z)$. For $f,\ g,\ h \in \mathbb{C}(S)^\times$, define

$$\eta(f,g,h) \;=\; \sum_{\substack{(a,b,c)=\sigma(f,g,h),\\ \sigma\in A_3}} \log|a| \left(\frac{1}{3}\, d\log|b|\, d\log|c| - d\arg(b)\, d\arg(c)\right),$$

where the sum is over all even (i.e., cyclic) permutations (a,b,c) of the ordered triple (f,g,h). The form is still closed, as

$$d\,\eta(f,g,h) \;=\; \mathrm{Re}\left[\frac{df}{f}\wedge\frac{dg}{g}\wedge\frac{dh}{h}\right],$$

and we may again compute the Mahler measure using this form. For instance, if we decide to use Jensen's formula with respect to z, we obtain (assuming for simplicity that P is monic as a polynomial in z):

$$\begin{aligned} m(P) &= \frac{1}{(2\pi)^2}\int_\Gamma \log|z|\, d\arg(x)\, d\arg(y) \\ &= -\,\frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z) \end{aligned} \tag{5.7}$$

where

$$\Gamma = \big\{(x,y,z)\in S\,:\,|x|=|y|=1,\ |z|\ge 1\big\}.$$

Paralleling the two-variable situation, η induces a map

$$\eta:\ \Lambda^3\big(\mathbb{C}(S)^\times\big)\otimes\mathbb{Q} \;\to\; \Omega^2(S).$$

As before, the goal then becomes to rewrite $\eta(x,y,z)$ as a linear combination of exact forms, all of a certain shape. The fact we will use is that

$$\eta(f,1-f,g) = d\,\omega(f,g)$$

where

$$\omega(f,g) = -D(f)\, d\arg(g) + \frac{1}{3}\log|g|\big(\log|1-f|\, d\log|f| - \log|f|\, d\log|1-f|\big).$$

So now if we can find a decomposition

$$x \wedge y \wedge z \;=\; \sum_j a_j \cdot x_j \wedge (1 - x_j) \wedge y_j \tag{5.8}$$

in $\Lambda^3\big(\mathbb{C}(S)^\times\big) \otimes \mathbb{Q}$, then $\eta(x, y, z)$ is exact, and it follows that

$$\begin{aligned} m(P) &= -\frac{1}{(2\pi)^2} \int_\Gamma \eta(x, y, z) \\ &= -\frac{1}{(2\pi)^2} \sum_j a_j \int_{\partial\Gamma} \omega(x_j, y_j). \end{aligned} \tag{5.9}$$

At this point, in contrast with the two-variable case, we still have one integral left to evaluate. We would like to use Stokes' Theorem again; unfortunately, there is no way the form $\omega = \sum a_j\, \omega(x_j, y_j)$ could be exact on S, as this would imply $\eta(x, y, z) = 0$. However, Maillot has proposed the following way of circumventing this difficulty. For points (x, y, z) on $\partial\Gamma = S \cap \{|x| = |y| = |z| = 1\}$,

$$0 = \overline{P(x, y, z)} = \overline{P}\big(x^{-1}, y^{-1}, z^{-1}\big),$$

so (x, y, z) also satisfies the reciprocal polynomial $P^\star(x, y, z)$. Thus

$$\partial\Gamma \;\subseteq\; C := \{P = 0\} \cap \{P^\star = 0\}.$$

C is a curve, since P is relatively prime to $P^\star$. Maillot's idea was that although ω cannot be exact on S, it might be exact on C.

Again, we wish to decompose into exact forms of a standard shape. The key fact here is:

$$\omega(f, f) = d\, P_3(f). \tag{5.10}$$

At this point, we should make some observations about the form $\omega(f,g)$. It is easily seen that

$$\omega(u, cv) = \omega(u, v), \quad \text{if } |c| = 1 \tag{5.11}$$

$$\omega(\bar{u}, \bar{v}) = \omega(u, v) \tag{5.12}$$

Also, in the second coordinate it is multiplicative:

$$\omega(u, v_1 v_2) = \omega(u, v_1) + \omega(u, v_2),$$

but in the first coordinate, it obeys the five-term relation

$$\omega(a, v) + \omega(b, v) + \omega\Big(\frac{1-a}{1-ab}, v\Big) + \omega(1-ab, v) + \omega\Big(\frac{1-b}{1-ab}, v\Big) \;=\; 0.$$

Hence, $\Lambda^2\big(\mathbb{C}(C)^\times\big) \otimes \mathbb{Q}$ does not accurately model the algebra of these forms. For the right setup, we make some definitions, more or less following Goncharov [10]. For an abelian group A, let $A_{\mathbb{Q}}$ denote $A \otimes \mathbb{Q}$. Let F be any field, and let

$$\mathcal{F}(F) = \mathbb{Z}[\mathbb{P}^1(F)] \,/\, \big\langle [0], [\infty] \big\rangle \;\cong\; \mathbb{Z}[F^\times].$$

For $x, y \in \mathbb{P}^1(F)$, define

$$r_2(x, y) = [x] + [y] + \Big[\frac{1-x}{1-xy}\Big] + [1-xy] + \Big[\frac{1-y}{1-xy}\Big],$$

if $x, y \neq \infty$ and $xy \neq 1$, and

$$r_2(x, \infty) = r_2(\infty, x) = r_2(x, 1/x) = [x] + [1/x],$$

thought of as elements of $\mathcal{F}(F)$. Let $R_2(F)$ be the subgroup generated by the elements $r_2(x, y)$ for all $x, y \in \mathbb{P}^1(F)$, and define

$$B_2(F) = \mathcal{F}(F) \,/\, R_2(F).$$

Then for $F = \mathbb{C}(C)$, $(B_2(F) \otimes F^\times)_\mathbb{Q}$ provides a natural setting to model calculations with $\omega(f,g)$. Our form $\omega = \sum_j a_j\, \omega(x_j, y_j)$ corresponds to

$$\sum_j a_j\, [x_j] \otimes y_j \;\in\; (B_2(F) \otimes F^\times)_\mathbb{Q}.$$

So using (5.10), if we can rewrite the above as

$$\sum_k b_k\, [z_k] \otimes z_k \tag{5.13}$$

for some elements $z_k \in \mathbb{C}(C)^\times$, then

$$\begin{aligned} m(P) &= -\frac{1}{(2\pi)^2} \int_{\partial\Gamma} \omega \\ &= -\frac{1}{(2\pi)^2} \sum_k b_k \left[P_3(z_k)\right]_{\partial\Gamma}, \end{aligned} \tag{5.14}$$

and the desired evaluation is obtained.

5.3 Connections with K-theory

As we explained in section 5.1, in the two-variable case, $m(P)$ may be evaluated by Stokes' Theorem if we can show that $\{x, y\} = 0$ in $K_2\big(\mathbb{C}(C)\big) \otimes \mathbb{Q}$. We find a similar situation in three-variable context; the possibility of evaluating $m(P)$ by Stokes' Theorem is equivalent to the vanishing of elements of certain K-groups.

We will first give a very brief overview of the relevant notions in algebraic K-theory. Our references for this material include [10], [11], [13], [18], [22] and [23]. Let R be an associative ring with unity. Historically, the groups $K_n(R)$ for $n = 0$, 1 and 2 were defined in a somewhat ad hoc fashion, motivated by topological K-theory. A number of definitions have been proposed to unify these constructions and extend them to $n \geq 3$, although not all of these definitions agree for higher n. Quillen constructs a topological space $BGL(R)^+$ that has the same homology as R, and then defines the abelian groups

$$K_n(R) = \pi_n\big(BGL(R)^+\big).$$

We will also make use of Milnor's definition. For any field F, define the tensor algebra (over $\mathbb{Z}$)

$$T(F) = \bigoplus_{n\geq 0} \big[\underbrace{F^\times \otimes \cdots \otimes F^\times}_{n \text{ times}}\big],$$

and let J be the ideal in $T(F)$ generated by elements of the form $x \otimes (1-x)$ for $x \neq 0, 1$. Then

$$K_*^M(F) = \bigoplus_{n\geq 0} K_n^M(F) = T(F)/J.$$

That is, $K_n^M(F)$ consists of the elements of $K_*^M(F)$ that are homogeneous of degree n. It may be shown that for $n \geq 2$,

$$K_n^M(F) = \Lambda^n\big(F^\times\big) \,/\, \big\langle x \wedge (1-x) \wedge y_3 \wedge \cdots \wedge y_n \big\rangle.$$

For $n = 0$, 1 and 2, $K_n(F) \cong K_n^M(F)$ (the isomorphism for $n = 2$ is a difficult theorem of Matsumoto). In particular, $K_0(F) \cong \mathbb{Z}$ and $K_1(F) \cong F^\times$.

For all n, there is a canonical homomorphism $K_n^M(F) \to K_n(F)$, which is injective up to torsion. We will speak of the injection $K_n^M(F)_\mathbb{Q} \hookrightarrow K_n(F)_\mathbb{Q}$ as if it were inclusion. The *indecomposable part* of $K_n(F)$ is defined to be

$$K_n^{ind}(F) = \operatorname{coker}\big(K_n^M(F) \to K_n(F)\big).$$

We may map $GL_n(F) \to GL_{n+1}(F)$ by sending a matrix $M \mapsto \left[\begin{smallmatrix} M & \\ & 1 \end{smallmatrix}\right]$. For all n, this gives maps

$$GL_n(F) \to GL(F) = \varinjlim GL_n(F).$$

From the Milnor-Moore theorem it follows that

$$K_n(F)_{\mathbb{Q}} = \operatorname{Prim} H_n\big(GL(F), \mathbb{Q}\big),$$

the "primitive part" of the homology. Further, by a theorem of Suslin, if F is an infinite field (which we shall henceforth assume), then $H_n\big(GL_n(F), \mathbb{Q}\big) = H_n\big(GL(F), \mathbb{Q}\big)$. Consequently, we obtain a filtration

$$K_n(F)_{\mathbb{Q}} = K_n^{(0)}(F)_{\mathbb{Q}} \;\supseteq\; K_n^{(1)}(F)_{\mathbb{Q}} \;\supseteq\; K_n^{(2)}(F)_{\mathbb{Q}} \;\supseteq\; \cdots,$$

by defining

$$K_n^{(i)}(F)_{\mathbb{Q}} = K_n(F)_{\mathbb{Q}} \cap \operatorname{image}\Big[H_n\big(GL_{n-i}(F), \mathbb{Q}\big) \to H_n\big(GL(F), \mathbb{Q}\big)\Big].$$

Define the quotients $K_n^{[i]}(F)_{\mathbb{Q}} = K_n^{(i)}(F)_{\mathbb{Q}} / K_n^{(i+1)}(F)_{\mathbb{Q}}$. Hence $K_n(F)_{\mathbb{Q}} = \oplus_{i \geq 0} K_n^{[i]}(F)_{\mathbb{Q}}$. It follows from a theorem of Suslin that

$$K_n^{[0]}(F)_{\mathbb{Q}} \cong K_n^M(F)_{\mathbb{Q}},$$

so $K_n^{ind}(F)_{\mathbb{Q}} = K_n^{(1)}(F)_{\mathbb{Q}}$.

Recall from section 5.2 that we defined the group $B_2(F)$ as essentially $\mathbb{Z}[F^\times]$ modulo relations satisfied by the dilogarithm; we will now do a similar construction for the trilogarithm. For x, y, $z \in F^\times$, define

$$\begin{aligned} r_3(x,y,z) = [-xyz] \ + \sum_{\substack{(a,b,c)=\sigma(x,y,z),\\ \sigma\in A_3}} \Bigg(& [ca-a+1] + \left[\frac{ca-a+1}{ca}\right] + [c] - [1] \\ & + \left[\frac{bc-c+1}{(ca-a+1)b}\right] - \left[\frac{ca-a+1}{c}\right] \\ & + \left[\frac{-(bc-c+1)}{ca-a+1}\right] - \left[\frac{bc-c+1}{(ca-a+1)bc}\right]\Bigg) \end{aligned}$$

in $\mathcal{F}(F)$, where the sum is over all even permutations (a,b,c) of (x,y,z). Define $R_3(F)$ to be the subgroup of $\mathcal{F}(F)$ generated by elements of the form:

- $r_3(x,y,z)$, except when $1 = x(1-z) = y(1-x) = z(1-y)$,
- $[x] - [x^{-1}]$, and
- $[x] + [1-x] + [1-x^{-1}] - [1]$,

and let $B_3(F) := \mathcal{F}(F)\,/\,R_3(F)$. We may define a map from $\mathcal{F}(\mathbb{C}) \to \mathbb{R}$ by sending $[z] \mapsto P_3(z)$ and extending by linearity. In fact, Goncharov showed

Theorem. *The trilogarithm $P_3(z)$ induces a well-defined map*

$$B_3(\mathbb{C}) \longrightarrow \mathbb{R}.$$

(Note that $r_3\big(1, X, \frac{1-X}{1-Y}\big)$ is actually the Spence-Kummer relation (2.9). Goncharov constructs the relations $r_3(x,y,z)$ geometrically, from cross-ratios of seven points in $\mathbb{P}^2(\mathbb{C})$, taken four at a time.)

So $B_3(\mathbb{C})$ and $B_2(\mathbb{C})$ are, in a sense, the natural domains for P_3 and $D = P_2$, respectively. The connection between these groups and K-theory is given by the following sequences of maps:

$$\mathfrak{B}_F(2) \ : \quad B_2(F)_{\mathbb{Q}} \xrightarrow{\delta_1^2} \Lambda^2\big(F^\times\big)_{\mathbb{Q}}$$

$$\mathfrak{B}_F(3) \ : \quad B_3(F)_{\mathbb{Q}} \xrightarrow{\delta_1^3} \big(B_2(F)\otimes F^\times\big)_{\mathbb{Q}} \xrightarrow{\delta_2^3} \Lambda^3\big(F^\times\big)_{\mathbb{Q}},$$

where

$$\delta_1^2([x]) = x\wedge(1-x), \quad \delta_1^3([x]) = [x]\otimes x, \quad \delta_2^3([x]\otimes y) = x\wedge(1-x)\wedge y,$$

and we take $\delta_1^2([1])$ and $\delta_2^3([1]\otimes y)$ both to be zero by convention. (The maps δ_1^2 and δ_2^3 may be defined prior to tensoring with $\mathbb{Q}$, but δ_1^3 is only well-defined modulo 6-torsion.) The kernel of the map from $B_2(F) \to \Lambda^2\big(F^\times\big)_{\mathbb{Q}}$ is known as the Bloch group[2] $\mathcal{B}(F)$. Note that $\mathcal{B}(F)_{\mathbb{Q}} = \ker(\delta_1^2)$.

It may be verified that $\mathfrak{B}_F(2)$ and $\mathfrak{B}_F(3)$ are actually complexes. When we specialize to the fields $E := \mathbb{C}(S)$ or $F := \mathbb{C}(C)$, the groups in the complexes model the various forms we have encountered, and the maps δ_i^j simply correspond to derivatives. Indeed, we have commutative diagrams

$$\begin{tikzcd}
B_2(F)_{\mathbb{Q}} \arrow[r, "\delta_1^2"] \arrow[d, "D"'] & \Lambda^2\big(F^\times\big)_{\mathbb{Q}} \arrow[d, "\eta"] \\
\Omega^0(C) \arrow[r, "d"] & \Omega^1(C)
\end{tikzcd}$$

and

$$\begin{tikzcd}
B_3(F)_{\mathbb{Q}} \arrow[r, "\delta_1^3"] \arrow[d, "P_3"'] & \big(B_2(F)\otimes F^\times\big)_{\mathbb{Q}} \arrow[d, "\omega"'] & \big(B_2(E)\otimes E^\times\big)_{\mathbb{Q}} \arrow[l, dotted] \arrow[r, "\delta_2^3"] \arrow[d, "\omega"'] & \Lambda^3\big(E^\times\big)_{\mathbb{Q}} \arrow[d, "\eta"'] \\
\Omega^0(C) \arrow[r, "d"] & \Omega^1(C) & \Omega^1(S) \arrow[l, dotted] \arrow[r, "d"] & \Omega^2(S)
\end{tikzcd}$$

[2]This definition follows Zagier and others; Goncharov defines the Bloch group differently.

(The dotted lines are because we do not have a true map there from E to F. Restriction of an element of E to C usually, but not always, gives an element of F; consider for instance $1/\,P^{\star}$.)

Let us examine the cohomologies of $\mathfrak{B}_F(2)$ and $\mathfrak{B}_F(3)$. (For both complexes, the leftmost group is considered to be in degree 1.)

Theorem.

$$\begin{array}{rcllll} H^2\big(\mathfrak{B}_F(2)\big) & \cong & K_2(F)_{\mathbb{Q}} & & & \textit{(Matsumoto)} \\ H^1\big(\mathfrak{B}_F(2)\big) & = & \mathcal{B}(F) & \cong & K_3^{ind}(F)_{\mathbb{Q}} & \textit{(Suslin)} \\ H^3\big(\mathfrak{B}_F(3)\big) & = & K_3^M(F)_{\mathbb{Q}} & \cong & K_3^{[0]}(F)_{\mathbb{Q}} & \textit{(Suslin)} \end{array}$$

These all fit the pattern $H^i\big(\mathfrak{B}_F(j)\big) \cong K_{2j-i}^{j-i}(F)_{\mathbb{Q}}$. In fact, we also have:

Conjecture (Goncharov).

$$H^2\big(\mathfrak{B}_F(3)\big) \cong K_4^{[1]}(F)_{\mathbb{Q}}$$
$$H^1\big(\mathfrak{B}_F(3)\big) \cong K_5^{[2]}(F)_{\mathbb{Q}}$$

As discussed at the end of section 5.1, the success of the algebraic method in the two-variable case is dependent on $x \wedge y$ lying in the image of δ_1^2, or equivalently, having $\{x, y\} = 0$ in $K_2\big(\mathbb{C}(C)\big)_{\mathbb{Q}}$. We now see that success in the three-variable case depends (at least conjecturally) on showing that the elements of $K_3^{[0]}\big(\mathbb{C}(S)\big)_{\mathbb{Q}}$ and $K_4^{[1]}\big(\mathbb{C}(C)\big)_{\mathbb{Q}}$ corresponding to $x \wedge y \wedge z$ and to the form ω from (5.9) are equal to zero. While this observation may not help us to do any calculations we couldn't do before, it provides a broad conceptual framework and provides a link to some powerful preexisting mathematics.

5.4 Revisiting the original polynomial

Recall from chapter 3 our first result:

$$m\big(1 + x + (1-x)(y+z)\big) = \frac{28}{5\pi^2}\,\zeta(3).$$

It would be interesting to prove this again using the techniques of the previous section; we will pursue this as far as we know how.

Recall from the proof of proposition 3.3 that we may do a change of variables, replacing the above polynomial with the Laurent polynomial

$$P(x,y,z) = 1 + x + (1-x)y(z - z^{-1})$$

without changing the Mahler measure. We let

$$P^\star(x,y,z) = xy \cdot P(x^{-1}, y^{-1}, z^{-1}) = (1+x)y + (1-x)(z - z^{-1});$$

this is relatively prime to P. Let S denote the surface $\{P = 0\}$. Thinking of x, y and z as elements of $\mathbb{C}(S)$,

$$y = \frac{-(1+x)}{(z-z^{-1})(1-x)}. \tag{5.15}$$

So in $\Lambda^3\big(\mathbb{C}(S)^\times\big) \otimes \mathbb{Q}$,

$$\begin{aligned} x \wedge y \wedge z \;&=\; x \wedge (1+x) \wedge z \;-\; x \wedge (1-x) \wedge z \;-\; x \wedge \left(\frac{z^2-1}{z}\right) \wedge z \\ &=\; (-x) \wedge (1+x) \wedge z \;-\; x \wedge (1-x) \wedge z \;+\; \frac{1}{2}\, z^2 \wedge (1-z^2) \wedge x. \end{aligned}$$

This is equal to $\delta_2^3(\Delta)$, for

$$\Delta \;=\; \big([-x] - [x]\big) \otimes z + \frac{1}{2}\,[z^2] \otimes x \quad \in \; \big(B_2\big(\mathbb{C}(C)\big) \otimes \mathbb{C}(S)^\times\big)_{\mathbb{Q}},$$

hence $\eta(x, y, z)$ is exact.

By (5.15), on the surface $S^\star : P^\star = 0$ we have

$$y^{-1} = \frac{-(1+x^{-1})}{(z^{-1}-z)(1-x^{-1})} = \frac{-(1+x)}{(z-z^{-1})(1-x)}.$$

Therefore on the curve $C := \{P = 0\} \cap \{P^\star = 0\}$,

$$1 = y \cdot y^{-1} = \left(\frac{-(1+x)}{(z-z^{-1})(1-x)}\right)^2,$$

so

$$\frac{1+x}{1-x} = \pm(z - z^{-1}), \tag{5.16}$$

hence $y = \mp 1$. The points on C having $y = 1$ biject with those having $y = -1$ via $(x, 1, z) \leftrightarrow (x, -1, -z)$. Let us restrict ourselves to those points with $y = -1$.

By the five-term relation, on $B_2\big(\mathbb{C}(C)\big)$ we have

$$[x] - [-x] = \left[\frac{1+x}{1-x}\right] - \left[\frac{x+1}{x-1}\right].$$

Also, from $(1+x)/(1-x) = z - z^{-1}$, it follows that

$$x = -\left(\frac{1-z+z^{-1}}{1+z-z^{-1}}\right).$$

Thus, observing that $[f] = -[1-f]$ in $B_2\big(\mathbb{C}(C)\big)$,

$$\begin{aligned}\Delta &= \big([-z+z^{-1}] - [z-z^{-1}]\big) \otimes z + \frac{1}{2}[z^2] \otimes \left(\frac{1-z+z^{-1}}{1+z-z^{-1}}\right) \\ &= \frac{1}{2}\big(-[1+z-z^{-1}] + [1-z+z^{-1}]\big) \otimes (z^2) + \frac{1}{2}[z^2] \otimes \left(\frac{1-z+z^{-1}}{1+z-z^{-1}}\right) \\ &= \frac{1}{2}\left\langle \alpha, z^2 \right\rangle - \frac{1}{2}\left\langle \beta, z^2 \right\rangle, \end{aligned} \tag{5.17}$$

where we define $\alpha := 1 - z + z^{-1},\ \beta := 1 + z - z^{-1},$ and

$$\langle f, g\rangle := [f] \otimes g + [g] \otimes f.$$

Further, for $|z| = 1$, $\beta = \bar{\alpha}$. So restricting to C, $\langle \beta, z^2\rangle$ corresponds to the form

$$\begin{aligned}
\omega(\beta, z^2) + \omega(z^2, \beta) &= -D(\beta)\, d\arg(z^2) - D(z^2)\, d\arg(\beta) \\
&= -D(\bar{\alpha})\, d\arg(z^2) - D(z^2)\, d\arg(\bar{\alpha}) \\
&= D(\alpha)\, d\arg(z^2) + D(z^2)\, d\arg(\alpha) \\
&= -\omega(\alpha, z^2) - \omega(z^2, \alpha),
\end{aligned}$$

which corresponds to $-\langle \alpha, z^2\rangle$. Hence it suffices to deal with $\Delta' = \langle \alpha, z^2\rangle$ in this calculation.[3] The symmetry and compactness of this form are appealing, but we still have to show that it lies in the image of δ_1^3.

Notice that

$$\alpha = \frac{-1}{z}(z^2 - z - 1) = \frac{(z - \varphi)(z + \varphi^{-1})}{-z},$$

once again showing that φ is seemingly unavoidable in this evaluation. We have searched for a way to deal with Δ' that bypasses φ but have not yet found one. Lalín has succeeded in completing the evaluation using the above factorization of α, although the proof is by no means simple. Incidentally, her evaluation, before simplifying, bears a striking similarity to the right side of equation (3.21), but interestingly, $\varphi^{\pm 3}$ does not seem to appear.

[3]However, Δ and Δ' are not equal as elements of $\big(B_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^\times\big)_{\mathbb{Q}}$. So if we are striving for as purely algebraic an approach as possible, this simplification might not be desirable.

In the process of trying to show that Δ' is in the image of δ_1^3, it seemed reasonable to try to find an analogue in this new language of one of our key tools in the original proof, the logarithmic form integration (3.10). As we mentioned earlier, in the remarks after equation (4.5), the arguments to the trilogarithms in that formula are precisely those used in the five-term relation. Motivated by that observation, we found the following relation, which may be of independent interest.

Proposition 5.1. *For any field F and a, $b \in F \setminus \{0,1\}$,*

$$\delta_1^3\Big([a]+[b]+\Big[\frac{1-b}{1-a}\Big]+\Big[\frac{a(1-b)}{b(1-a)}\Big]-\Big[\frac{a}{b}\Big]\Big)$$
$$= \big\langle a,b\big\rangle \;+\; \Big\langle \frac{1-b}{1-a},\frac{a(1-b)}{b(1-a)}\Big\rangle.$$

Proof. We will use the shorthand $u=\frac{1-b}{1-a}$. The left side of the equation in the proposition is

$$\begin{aligned}
&[a]\otimes a+[b]\otimes b+[u]\otimes u+\Big[\frac{au}{b}\Big]\otimes\Big(\frac{au}{b}\Big)-\Big[\frac{a}{b}\Big]\otimes\frac{a}{b}\\
&\quad=\Big([a]+\Big[\frac{au}{b}\Big]-\Big[\frac{a}{b}\Big]\Big)\otimes a+\Big([b]-\Big[\frac{au}{b}\Big]+\Big[\frac{a}{b}\Big]\Big)\otimes b+\Big([u]+\Big[\frac{au}{b}\Big]\Big)\otimes u.
\end{aligned}$$

Since

$$r_2(1-a,u) \;=\; -[a]+[b]+[u]-\Big[\frac{au}{b}\Big]+\Big[\frac{a}{b}\Big]$$

is equal to zero in $B_2(F)$, the above becomes

$$\begin{aligned}
&= \big([b]+[u]\big)\otimes a+\big([a]-[u]\big)\otimes b+[u]\otimes u+\Big[\frac{au}{b}\Big]\otimes u\\
&= [b]\otimes a+[a]\otimes b+[u]\otimes\Big(\frac{au}{b}\Big)+\Big[\frac{au}{b}\Big]\otimes u,
\end{aligned}$$

as claimed. □

We would hope to be able to use the proposition to show that Δ or Δ' lies in the image of δ_1^3. Unfortunately, the right side of the proposition involves a pair of $\langle\cdot,\cdot\rangle$'s, so using it to eliminate one $\langle\cdot,\cdot\rangle$ only causes another to appear. And while Δ does have two $\langle\cdot,\cdot\rangle$'s, it does not appear to be possible to use one to eliminate the other.

It might be hoped that iterating the proposition would yield new results, e.g. by taking $a' = \frac{1-b}{1-a}$ and $b' = \frac{a(1-b)}{b(1-a)}$. However, the map on $(F\backslash\{0,1\})^2$

$$\sigma : (a,b) \longmapsto \Big(\frac{1-b}{1-a}, \frac{a(1-b)}{b(1-a)}\Big)$$

is of order four; indeed, $\sigma^2(a,b) = (1/a, 1/b)$. So since $[1/a] \otimes (1/b) = [a] \otimes b$, σ induces an involution on our forms $\langle\cdot,\cdot\rangle$, which prevents us from obtaining anything new from iteration.

Bibliography

[1] S. Bloch. Function theory of polylogarithms. In *Structural properties of polylogarithms*, volume 37 of *Math. Surveys Monogr.*, pages 275–285. Amer. Math. Soc., Providence, RI, 1991.

[2] D. W. Boyd. Speculations concerning the range of Mahler's measure. *Canad. Math. Bull.*, 24(4):453–469, 1981.

[3] D. W. Boyd. Mahler's measure and special values of L-functions. *Experiment. Math.*, 7(1):37–82, 1998.

[4] D. W. Boyd. Uniform approximation to Mahler's measure in several variables. *Canad. Math. Bull.*, 41(1):125–128, 1998.

[5] D. W. Boyd and F. Rodriguez Víllegas. Mahler's measure and the dilogarithm (I). *Canad. J. Math.*, 54(3):468–492, 2002.

[6] D. W. Boyd, F. Rodriguez Víllegas, and N. M. Dunfield. Mahler's measure and the dilogarithm (II). Preprint, 2003.

[7] C. Deninger. Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbf{Z}^n$-actions. *J. Amer. Math. Soc.*, 10(2):259–281, 1997.

[8] H. Gangl. Families of functional equations for polylogarithms. In *Algebraic K-theory (Poznań, 1995)*, volume 199 of *Contemp. Math.*, pages 83–105. Amer. Math. Soc., Providence, RI, 1996.

[9] H. Gangl. Functional equations for higher logarithms. *Selecta Math. (N.S.)*, 9(3):361–377, 2003.

[10] A. B. Goncharov. Geometry of configurations, polylogarithms, and motivic cohomology. *Adv. Math.*, 114(2):197–318, 1995.

[11] M. Kolster. K-theory and arithmetic. CICMA Report 2000-08, Concordia University, Montreal, 2000.

[12] M. Lalín. Some examples of Mahler measures as multiple polylogarithms. *J. Number Theory*, 103(1):85–108, 2003.

[13] M. Lalín. An algebraic integration for Mahler measure. Preprint, 2004.

[14] D. H. Lehmer. Factorization of certain cyclotomic functions. *Ann. of Math. (2)*, 34:461–479, 1933.

[15] L. Lewin. *Polylogarithms and associated functions*. North Holland, 1981.

[16] K. Mahler. On some inequalities for polynomials in several variables. *J. London Math. Soc. (2)*, 37:341–344, 1962.

[17] V. Maillot. Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables. *Mém. Soc. Math. Fr. (N.S.)*, (80):vi+129, 2000.

[18] J. Milnor. Algebraic K-theory and quadratic forms. *Invent. Math.*, 9:318–344, 1969/1970.

[19] D. Ramakrishnan. On the monodromy of higher logarithms. *Proc. Amer. Math. Soc.*, 85(4):596–599, 1982.

[20] H. F. Sandham. A logarithmic transcendent. *J. London Math. Soc.*, 24:83–91, 1949.

[21] C. J. Smyth. On measures of polynomials in several variables. *Bull. Austral. Math. Soc.*, 23:49–63, 1981.

[22] A. A. Suslin. Homology of GL_n, characteristic classes and Milnor K-theory. In *Algebraic K-theory, number theory, geometry and analysis (Bielefeld, 1982)*, volume 1046 of *Lecture Notes in Math.*, pages 357–375. Springer, Berlin, 1984.

[23] A. A. Suslin. Algebraic K-theory of fields. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 222–244, Providence, RI, 1987. Amer. Math. Soc.

[24] S. Vandervelde. A formula for the Mahler measure of $axy + bx + cy + d$. *J. Number Theory*, 100(1):184–202, 2003.

[25] F. Rodriguez Víllegas. Modular Mahler measures (I). In *Topics in number theory (University Park, PA, 1997)*, volume 467 of *Math. Appl.*, pages 17–48. Kluwer Acad. Publ., Dordrecht, 1999.

[26] D. Zagier. The Bloch-Wigner-Ramakrishnan polylogarithm function. *Math. Ann.*, 286(1-3):613–624, 1990.

[27] D. Zagier. Polylogarithms, Dedekind zeta functions and the algebraic K-theory of fields. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 391–430. Birkhäuser Boston, Boston, MA, 1991.

Vita

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